ENGME 712
Applied Mathematics in Mechanics

Turibius Rozario
Instructor: Douglas Holmes

September 2, 2025 to December 9, 2025

Holmes developed this class. It allows us to use all the math tools to figure out how to use it in our field. This course tackles the challenge of what to do when we can't solve a problem. A lot of unsolved problems use iterations, but is not analytical. 3 Body Problem is an example.

Peterbation is 'adding in' the hard part of the equation one at a time, to create a new approximation. Is used in fluid mechanics, solid mechanics, and other problems where system of equations cannot be solved analytically.

Linear stability analysis for things such as boiling and freezing that create large changes in system characteristics.

Variational tools allow optimizing minima or maxima. Variational calculus is essential for robotics, path planning, and other constrained optimization.

We can mathematically derive conservation laws.

Differential geometry is for curved geometry, such as path planning around curves, and soft robotics.

Homework are typically 4 or 5 problems; submitted to GradeScope. Once submitted, we present how we approach the work. If we don't get a problem right, we can resubmit and have it re-graded. Holmes is more interested in our process than problem.

The midterm example is just one large practice problem. In the project, we pick a paper that uses several of the applied mathematical techniques, and then teach our fellow classmates.

Two homework assignment are posted: one of them is how to download and install Mathematica (includes some problems and how to use Mathematica); homework 1 is due next Tuesday.

His website is on online. Applied Mathematics by J.D. Logan is a fun book. Scaling by G.I. Barenblatt is useful for the first topic. For a book to follow along with is Introduction to the Foundations of Applied Mathematics by Mark Holmes. For Fluid Mechanics, Perturbation Methods by E.J. Hinch is good. Get and read the book Nonlinear Dynamics and Chaos by Strogatz since it is simply well-written.

Problem sets are homework, and papers are more so readings. The readings are on Blackboard, and questions on gradescope.

1 Dimensional Analysis

In any problem, the dimensions have to match. We are not concerned about the exact unit, but what it measures, that is, length instead of meters. [L] , [T] , [M] , [θ] , [I] , [N] for length, time, mass, temperature, current, luminosity, and mole (amount of stuff), respectively. These are completely unrelated to each other; they are all independent from each other. Every other unit is based of the 7 fundamentals, and can have different ways of being represented, such as density being mass density or quantity density.

Burning Candle

Burning candle problem questions:

- Diameter of candle
- Chemical composition
- Width of wick
- Atmospheric pressure
- Room size / temperature

We have to decide which questions are important. The candle diameter is the most important. We want to start the model with as little variables as possible, and ignore minor contributions. We try to make other variables be identical. We can make two questions:

- How much energy is in the candle?
- What is the rate at which energy is dissipitated through light and heat?

There will be some amount of chemical energy density, $u_d = Energy/Volume = \frac{ML^2}{T^2} \times \frac{1}{L^3} = \frac{M}{T^2 L}$.

The next thing we want to get dimensions of is the rate of energy dissipation: $P = Energy/Time = \frac{ML^2}{T^3}$.

We can do a balance of equations. We say that the two candles have identical volume V:

$$u_d V = PT$$

These are both energy. Here, we will be concerned about the velocity of the candle: the height change per time.

$$\begin{aligned}
v &= H/T \\
V &= \pi R^2 H \\
u_d V &= PT \\
\Rightarrow u_d(\pi R^2 H) &= PT \\
\Rightarrow H/T &= P/(u_d \pi R^2) \\
\Rightarrow v &= \frac{P}{u_d \pi R^2}
\end{aligned}$$

Sometimes variables can be tucked away in another. Such as in a diffusion equation for the candle would tuck away the radius.

This should give us everything we need for homework 1.

Height of a Projectile

Things that might matter for a projectile thrown straight up are, for it's max height x_{max}

- mass m
- initial velocity v_i
- gravity g

The units of the projectile height should be some product of the above things that we state that matters:

$$[\mathrm{x_{max}}] = \left[\mathrm{m^a v_i^b g^c}\right]$$

where a, b, c are unknown. We place the fundamental dimensions:

$$L = M^a (L/T)^b (L/T^2)^c$$
$$L = M^a L^{b+c} \frac{1}{T^{b+2c}}$$

We can now equate all the exponents:

$$\begin{aligned}
[L]: \quad & 1 & = b + c \\
[M]: \quad & 0 & = a \\
[T]: \quad & 0 & = -b - 2c
\end{aligned}$$

Notice that the value of a is zero; the math tells us what doesn't matter. If we solved for the other two, we get $b = 2, c = -1$.

Mathematical symbol for scaling is a tilde:

$$x_{\max} \sim \frac{v_i^2}{g}$$

or can be done using some constant

$$x_{\max} = \alpha \frac{v_i^2}{g}$$

Ideally, α should be a numerical number on the order of 1. This brings us to a model that can be tested. This constant can be consistent drag, geometry of an object, etc.

By doing the mathematics, we can determine if a problem has any major discrepancy.

Toppling Dominoes

What is the velocity of the wave front as a function of some variables?

Velocity would be length over time. Things that can factor:

- Air density
- Gravity*
- Mass
- Height*
- Length between different dominoes*
- Initial toss / impulse
- Friction

Scaling law emerges during steady state once it gets going. We can ignite a candle very fast or slow, but it will very quickly burn at a steady state.

$$
\begin{aligned}
[V] &= [d^a h^b g^c] \\
\frac{L}{T} &= L^a L^b (\frac{L}{T^2})^c \\
\Rightarrow c &= 1/2 \\
\Rightarrow a &= 1 - b - 1/2 \\
\Rightarrow v &\sim \sqrt{dg} F_1(h/d) \\
\Rightarrow b &= 1 - 1 - 1/2 \\
\Rightarrow v &\sim \sqrt{hg} F_2(d/h)
\end{aligned}
$$

The h/d and d/h is a dimensionless Π, since they are unit-less. We are required to find what the F_1 and F_2 is, but if we ignore it, we can still predict v pretty accurately.

Drag on a sphere

Suppose we want to know the drag force on a sphere in a fluid, with variables $R, \mu, \vec{v}, \rho$. We know the dimensions of force, and that the variables must multiply to give us such dimensions.

$$
\begin{aligned}
F_D &= f(R, v, \rho, \mu) \\
F_D &= R^a v^b \rho^c \mu^d \\
\frac{ML}{T^2} &= L^a (\frac{L}{T})^b (\frac{M}{L^3})^c (\frac{M}{LT})^d
\end{aligned}
$$

Here, we have more unknown variables a, b, c, d than independent variables M, L, T. Doing b:

$$
L: \qquad 1 \qquad = a + b
$$

Knowing the Pi are important since physical systems can maintain physical similarity as we scale up or down. We can make scale models: if we understand the fundamental mechanics enough, and the physics are not changing, then we can test something large by testing a smaller size.

This will be valid for when physics does not change. But it can change, such as mach number. Check his website for scaling. We will have to make a new equation be a hypothesis, and only holds true for what we test on. Search up 'incomplete similarities'. Fatigue testing is an example where the physics does not scale: for fatigue testing, scaling does not work since the inherent material lengths are un-scalable. We do a hypothesis, choose a regime, test on simulation, test on real life, and see if the two tests matches.

We list the dimensionless Pi in its own function since there could be a cos or something tucked away inside. We can discard Pi groups thate are too large > 10 or too small $\Pi < 0.1$.

1.1 Similarity Variables

There are not a lot of PDEs in our undergraduate, since they often do not have a closed-form analytical solution. Sometimes we can turn it into ODEs.

In our course, ordinary derivatives are $\frac{d}{dx}()$, partials are $\frac{\partial}{\partial x}()$, and there is one more special $\delta()$.

Similarity variables aid us in reducing the dimensions, say, 2 variables into 1. An example is 1D diffusion:

$$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} \tag{1}$$

u is the concentration. We can have boundary conditions. For example, at the start of a bar, we have $u(0,t) = u_0$, an initial concentration. Infinitely far away, the concentration is $u(\infty, t) = 0$. Then, we have an initial conditution, such as $u(x, 0) = 0$.

Here, suppose we do not know what D is. We know for a fact that the units on the left and right must match. We know that u is the concentration. When we take a derivative of a funcction that has a particular dimension, say, $f(x) = x^3$ where $[x] = L$, then $[f(x)] = L^3$. If we take the spatial derivative, $\frac{df(x)}{dx}$, then we will get L^2. The dot, $\dot{x}$ typically almost always mean time derivative. The apostrophe is an ordinary derivative, x', which is typically $\frac{df}{dx}$. Dynamics very often uses the time derivative. Then there is $f_{,xx} = \frac{\partial^2 f}{\partial x^2}$; the professor often uses $\partial_{xx} f$ instead, or $\partial_x^2 f$.

In the diffusion equation, the independent variables are x, t, and the parameters are D, u_0 . A solution to this should be given by

$$u = f(x, t; D, u_0)$$

We can list the dimensions:

$$
\begin{aligned}
[u] &= x^a t^b D^c u_0^d \\
ML^{-3} &= L^a T^b (\frac{L^2}{T})^c (\frac{M}{L^3})^d \\
\vdots &= \vdots \\
u &= u_0 f \left(\underbrace{\frac{x}{\sqrt{Dt}}}_{\eta} \right)
\end{aligned}
$$

This tells us that u is dependent on a combination effect of the independent variables. We call it eta η in this case. We can do

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= u_0 f'(\eta) \frac{\partial \eta}{\partial t} \\
&= u_0 f'(\eta) \frac{\partial}{\partial t} (\frac{x}{\sqrt{Dt}}) \\
&= u_0 f'(\eta) \times - \frac{x}{2\sqrt{D} t^{3/2}} \\
&= -u_0 f'(\eta) \underbrace{\frac{x}{\sqrt{Dt}}}_{\eta} \frac{1}{2t}
\end{aligned}
$$

We can take another derivative:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &= u_0 f''(\eta) \left(\frac{\partial \eta}{\partial x} \right)^2 + u_0 f'(\eta) \cancelto{0}{\frac{\partial^2 \eta}{\partial x^2}} \\
\frac{\partial \eta}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{x}{\sqrt{Dt}} \right) \\
&= \frac{1}{\sqrt{Dt}} \\
D \left(u_0 f''(\eta) \frac{1}{Dt} \right) &= -u_0 f'(\eta) \frac{\eta}{2t} \\
\Rightarrow f''(\eta) &= -\frac{1}{2} \eta f'(\eta)
\end{aligned}
$$

We can obtain the solution of the last equation above to obtain $f(\eta)$, and then finally solve $u = u_0 f \left(\frac{x}{\sqrt{Dt}} \right)$.

When $x = 0$, $\eta = 0$. When $x = \infty, \rightarrow \eta = \infty$. When $t = 0, \rightarrow \eta = \infty$. The last two boundary conditions merge. We have the conditions:

$$\begin{aligned} u(0,t) &= u_0 f(0) && = u_0 \\ u(\infty t) &= u_0 f(\infty) && = 0 \\ u(x,0) &= u_0 f(\infty) && = 0 \end{aligned}$$

This tells us that $f(0) = 1$, and $f(\infty) = 0$. When we reduce the number of variables in the equations, we reduce the number of boundary conditions that are available. In mathematica, the problem we try to solve is:

$$f''(\eta) = -\frac{1}{2}\eta f'(\eta) \tag{2}$$

$$f(0) = 1 \tag{3}$$

$$f(\infty) = 0 \tag{4}$$

If we do a log-log plot of our dependent and independent variable, say $x \sim T^{\alpha}$, we would get a straight line with different slopes depending on what α. A pre-factor is a constant that shifts the line, but we can see that the scaling is same due to the slope, and therefore, adjust the pre-factor accordingly.

Overall steps in Dimensional Analysis

1. Create testable hypothesis
2. Identify the important parameters based on intuition, boundary conditions, initial conditions, simulation/experiments, etc. Use the governing equations.
3. Make predictions
4. Identify the dimensions
5. Using similarity variables, turn PDEs to ODEs

DA is one of the ways to solve PDEs by converting it to ODEs.

2 Non-dimensionalization

Dimensionless groups, i.e., Pi groups, are ratios. Most powerful way to solve equation we can't solve is by throwing away as many terms as possible. Things like Laplace transform, Fourier transform, etc. simplify the equations. Non-dimensionalization is also a good initial step to solving problems: by doing this, we can take unknown problems that we don't know how to solve, and make it similar to another problem that was solved. As an example, Reynolds number

$$\mathrm{Re} = \frac{Rv\rho}{\mu} \tag{5}$$

helps identify different states. Small dimensionless parameters are very helpful for approximating non-linear problems.

Projectile in gravitational field

When approximating equations, we want to know what is supposed to big vs small. When we non-dimensionalize, we make problems similar. Such as instead of looking at length only, looking at the ratio of two lengths.

$$\ddot{x} = \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -\frac{gR^2}{(R + x(t))^2} \tag{6}$$

We can set initial values, the initial velocity and location: $x(0) = 0$ and $\dot{x}(0) = v_0$. Here, the R, v_0, g are parameters, t is independent variable, and x are dependent variable.

To non-dimensionalize this, we introduce change of variable first: let's find the two variables in the problem that we can replace. Suppose dimensionless time is s , such that $t = t_c s$. The t_c is the characteristic time; the characteristic time could be the period for example, and we'll discover the special trait as we continue solving the problem.

For the height, we can non-dimensionalize that as $x = x_c u$. We can now do

$$\begin{aligned}
x_c u(0) &= 0 \\
\Rightarrow u(0) &= 0 \\
\frac{\mathrm{d}(x_c u(0))}{\mathrm{d}(t_s s)} &= v_0 \\
\Rightarrow \frac{\mathrm{d}u(0)}{\mathrm{d}s} &= \frac{v_0 t_c}{x_c} \\
\frac{\mathrm{d}^2(x_c u)}{\mathrm{d}(t_c s)^2} &= -\frac{gR^2}{(R + x_c u)^2} \\
\Rightarrow \underbrace{\frac{x_c}{t_c^2}}_{\text{acceleration}} \times \frac{\mathrm{d}^2 u}{\mathrm{d}s^2} &= \frac{-gR^2}{(R + x_c u)^2}
\end{aligned}$$

We now want to collect the parameters into dimensionless groups. We see that one side has an acceleration term, which can be turned non-dimensional by dividing both sides with g:

$$\begin{aligned}
\underbrace{\frac{x_c}{g t_c^2}}_{\Pi_1} \cdot \frac{\mathrm{d}^2 u}{\mathrm{d}s^2} &= -\frac{R^2}{(R + x_c u)^2} \\
&= -\frac{1}{(1 + u \underbrace{\frac{x_c}{R}}_{\Pi_2})^2} \\
\frac{\mathrm{d}u(0)}{\mathrm{d}s} &= \underbrace{\frac{v_0 t_c}{x_c}}_{\Pi_3}
\end{aligned}$$

As a sanity check, ensure that:

1. Π groups contain parameters and not variables
2. They are dimensionless
3. The Π group are independent, as in, they cannot be written in terms of other Π groups.

Now, we want to determine what the characteristic values t_c and x_c are. Again, all the Π groups here are

$$\Pi_1 = \frac{x_c}{gt_c^2}$$
$$\Pi_2 = \frac{x_c}{R}$$
$$\Pi_3 = \frac{v_0 t_c}{x_c}$$

We want to see which Π groups have more importance. We set one of the groups to be 0 to see what messes up the original equation. If $\Pi_1 = 0$, then our initial conditions fail, so Π_1 is essential. Π_3 is also essential since if it is 0, v_0 is zero, and we are not moving at all.

Always choose Π that are in initial and boundary condition. Additionally, in general, we should choose a Π to be essential such that, if 0, changes the problem.

Since we know what Π are important, we set them to be 1 so that they don't disappear. We have two Π that are set to 1, and 2 unknowns. Setting them to 1, we get

$$x_c = \frac{v_0^2}{g}$$
$$t_c = \frac{v_0}{g}$$

and in a lab, we would plot (or do experiment with) xg/v_0^2 w.r.t. tg/v_0.

The contribution of v_0 is very small relative to R, and consequently, does not contribute much. So in

$$\frac{\mathrm{d}^2 u}{\mathrm{d}s^2} = -\frac{1}{(1 + \frac{v_0^2}{gR} u)^2} = -\frac{1}{(1 + \epsilon u)^2} \tag{7}$$

Non-dimensionalizing Pendulum

$$l \frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + k \frac{\mathrm{d}\theta}{\mathrm{d}t} + g \sin\theta = 0 \tag{8}$$

where the initial condition is $\theta(0)0 = \theta_0$ and $\dot{\theta}(0) = \omega_0$. We can set non-dimensional variables $t = t_c s$ and $\theta = \theta_c \phi$.

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}(\theta_c\phi)}{\mathrm{d}(t_c s)}$$
$$\frac{\mathrm{d}\phi(0)}{\mathrm{d}s} = \underbrace{\frac{t_c}{\theta_c}\omega_0}_{\Pi_4}$$
$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{\theta_c}{t_c^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}$$
$$\phi(0) = \underbrace{\frac{\theta_0}{\theta_c}}_{\Pi_3}$$

We can set Π_3 and Π_4 to be 1 to obtain the characteristic terms to be $\theta_c = \theta_0$ and $t_c = \theta_0/\omega_0$. We can now write the original differential equation as

$$\Rightarrow \frac{\theta_c}{t_c^2}\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{k}{l} \times \frac{\theta_c}{t_c} \times \frac{\mathrm{d}\phi}{\mathrm{d}s} + \frac{g}{l}\sin(\theta_c\phi)0 = 0$$
$$\Rightarrow \frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \underbrace{\frac{kt_c}{l}}_{\Pi_1} \times \frac{\mathrm{d}\phi}{\mathrm{d}s} + \underbrace{\frac{gt_c^2}{l\theta_c}}_{\Pi_2}\sin(\theta_c\phi)0 = 0$$
$$\Rightarrow \frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \epsilon\frac{\mathrm{d}\phi}{\mathrm{d}s} + \lambda\sin(\theta_0\phi) = 0$$

Suppose we set $\theta_0 = \pi/4$, $\omega_0 = 1\frac{rad}{s}$, $l = 1\,\mathrm{m}$, and $g = 10\,\mathrm{m/s^2}$. Then, we can see that if k is a small value, the ϵ term is very small; with $\lambda = 8$ unchanged, the equation is as if there is hardly any damping, like in air. If, however, the ϵ term is very large such as $k = 100\,\mathrm{m/s}$, then the equation lacks harmonics, as if it is in honey.

Nonlinear diffusion equation

Kolmogorov-Petrovsky-Piskunov (KPP/Fisher equation), also the reaction-diffusion equation. Reaction diffusion model appears everywhere, and has the form

$$\underbrace{D\frac{\partial^2 c}{\partial x^2} = \frac{\partial c}{\partial t}}_{\text{diffusion equation}} - \lambda(\gamma - c)c \tag{9}$$

This equation is a non-linear reaction diffusion equation. Boundary condition is $c(0,t) = 0$ and $c(l,t) = 0$. The initial conditions is $c(x,0) = c_0\sin(5\pi\frac{x}{l})$.

In a regular diffusion equation, concentration smooths out over term. Here, there is a wave front that propagates and diffuses at the same time. Two limits:

1. Weak growth rate / nonlinearity
2. Weak diffusion

Self-similar and similarity variable mean the same thing.

3 Perturbation Methods: Regular

We start with a hard problem, remove the hard thing, solve the easy thing, and add back the hard thing. This is similar to using the ϵ to make terms derivative. In the non-dimensionalized projectile problem, we have

$$\ddot{u} = -\frac{1}{(1+\epsilon u)^2}$$

where $\dot{u}(0) = 1$ and $u(0) = 0$. We will begin approximating. One of the easiest approximation is Taylor series, which yields:

$$\ddot{u}(s) = -1 + 2\epsilon u(s) - 3\epsilon^2 u^2(s) + O(\epsilon^3)$$

Note that we can claim equality so long as we include $O(\cdot)$. Next step is to assume a solution in the form. In this case, we expect the solution to be in the form

$$u(s) \approx u_0(s) + \epsilon u_1(s) + \epsilon^2 u_2(s) + \ldots$$

where we need to identify $u_i(s)\, s.t.\, i = 1, 2, \ldots$. We know the value of ϵ since we run the experiments. We can also do this for its solutions:

$$\dot{u}(s) \approx \dot{u}_0(s) + \epsilon \dot{u}_1(s) + \epsilon^2 \dot{u}_2(s) + \ldots$$

and similarly:

$$\ddot{u}(s) \approx \ddot{u}_0(s) + \epsilon \ddot{u}_1(s) + \epsilon^2 \ddot{u}_2(s) + \ldots$$

Perturbations can be regular or singular. Regular is simpler. To do singular, we have to do change of variables, and then regular.

We will insert the last three approximate equations into the original Taylor series:

$$\begin{aligned}
\ddot{u}_0(s) + \epsilon \ddot{u}_1(s) + \epsilon^2 \ddot{u}_2(s) + \ldots &= -1 \\
&\quad + 2\epsilon(u_0(s) + \epsilon u_1(s) + \epsilon^2 u_2(s) \ldots) \\
&\quad - 3\epsilon^2(u_0(s) + \epsilon u_1(s) + \epsilon^2 u_2(s) + \ldots)^2 \\
&\quad + \ldots \\
\dot{u}_0(0) + \epsilon \dot{u}_1(0) + \epsilon^2 \dot{u}_2(0) + \ldots &= 1 \\
u_0(0) + \epsilon u_1(0) + \epsilon^2 u_2(0) + \ldots &= 0
\end{aligned}$$

This is an iterative process now. We set $O(1) \to \epsilon = 0$, which allows us to cancel all the terms above except the numerical values:

$$\begin{aligned}
\ddot{u}_0(s) &= -1 \\
\dot{u}_0(0) &= 1 \\
u_0(0) &= 0
\end{aligned}$$

We integrate this twice and include the initial conditions, which yields

$$u_0(s) = \frac{1}{2}(2s - s^2)$$

Next, we solve for terms with ϵ: $O(\epsilon) \rightarrow \epsilon = 1$:

$$\begin{aligned}
\cancel{\epsilon}\ddot{u}_1(s) &= 2\cancel{\epsilon}u_0(s) \\
&= 2s - s^2 \\
\cancel{\epsilon}\dot{u}_1(0) &= 0 \\
\cancel{\epsilon}u_1(0) &= 0
\end{aligned}$$

which can be solved to yield

$$u_1(s) = \frac{1}{12}(4s^3 - s^4)$$

Similarly, for u_2 and its derivatives: $O(\epsilon^2)$:

$$\begin{aligned}
\cancel{\epsilon}^2\ddot{u}_2(s) &= 2\cancel{\epsilon}^2 u_1(s) - 3\cancel{\epsilon}^2(u_0(s))^2 \\
\cancel{\epsilon}^2\dot{u}_2(0) &= 0 \\
\cancel{\epsilon}^2 u_2(0) &= 0
\end{aligned}$$

which solves out to be

$$u_2(s) = \frac{1}{360}(-90s^4 + 66s^5 - 11s^6)$$

As a general rule of thumb, we should take up to $O(\epsilon)$ if $\epsilon < 0.01$, $O(\epsilon^2)$ if $\epsilon \approx 0.1$, and $O(\epsilon^3)$ if $\epsilon \approx 0.5$.

Any problem where the ϵ multiplies the highest order derivative or highest order polynomial (in a polynomial equation) requires the singular perturbation. Otherwise, such as in the above equations, it requires a regular perturbation method.

Let

$$\begin{aligned}
\ddot{v}(t) + \epsilon v(t)^2 + v(t) &= 0 \\
v(0) &= 1 \\
\dot{v}(0) &= 0 \\
v(t) &\approx v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + \ldots \\
\dot{v}(t) &\approx \dot{v}_0(t) + \epsilon \dot{v}_1(t) + \epsilon^2 \dot{v}_2(t) + \ldots \\
\ddot{v}(t) &\approx \ddot{v}_0(t) + \epsilon \ddot{v}_1(t) + \epsilon^2 \ddot{v}_2(t) + \ldots
\end{aligned}$$

4 Perturbation Methods: Single

A lot of problems in mechanics are singular problems. A singular perturbation problem occurs if the small parameter ϵ multiplies the highest order derivative; for example:

$$\epsilon w^{(4)}(x) + w''(x) = 0$$

which is the beam equation. Alternatively, ϵ could multiply the highest order polynomial

$$\epsilon x^5 + x^3 - 2x + x = 0$$

which may show up if we try to do a transformation such as Fourier transform and eigenvalue problems. The presence of such an ϵ signifies the existence of a boundary layer.

A shell turned inside out has a slight curve at the lip. The boundary layer is a minimum in this type of shell. A less curved shell, if inverted, is harder to stay inverted.

Moss exists only in a boundary layer of a fluid, due to dew forming.

If we try to do regular perturbation method with $\epsilon = 0$ for order 1 solution, we would lose the boundaries. For example, in the beam equation, we'd go from 4 boundary conditions to 2 if $\epsilon = 0$.

We can simply solve away from the boundary layer, then in the boundary layer, and match the two solutions asymptotically. Approach:

1. Split the problem: Regular perturbation, referred to as 'outer' solution.
2. Rescale the equation to stretch the independent variable in the neighborhood of the boundary layer. This involves
   (a) rescale by change of variable. If we had a function $y(x)$, we'd rescale it as $\bar{y}(\bar{x})$ where $\bar{x} = \frac{x - x_{bl}}{\epsilon^\gamma}$. Typically we'd know where the boundary layer x_{bl} is, and ϵ^γ is determined by the method of dominant balance.
   (b) insert the change of variable into the equations. This means inserting the $\bar{y}(\bar{x})$ into the equation, including any derivatives. For example, if $x_{bl} = 0$, we can say $x = \epsilon^\gamma \bar{x}$, and
   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\epsilon^\gamma} \frac{\mathrm{d}\bar{y}}{\mathrm{d}\bar{x}}$$
   (c) find a consistent balance; this involves multiplying one of the solution with some constant γ. We balance the exponent of the important term that was lost when $\epsilon = 0$ with each other term. Each time we do this, we find a potential γ . Finding the balance involves wanting the important term to remain dominant as $\epsilon \to 0$.
   (d) solve the resulting regular perturbation, known as the 'inner' solution
3. Match the inner and outer solution to obtain the 'composite' solution. We are sort of taking the limit between the boundary layer solution 'leaving the boundary layer' and the non-boundary layer solution 'entering the boundary layer'. This involves matched asymptotic analysis, and can be written as
   $$\lim_{\bar{x} \to \text{exiting BL}} \bar{y}(\bar{x}) = \lim_{x \to \text{entering BL}} y(x)$$

$$\begin{aligned} \epsilon y''(x) + y'(x) + 2y(x) &= 0 \qquad\qquad s.t.\, 0 \leq x \leq 1 \\ y(0) &= 0 \\ y(1) &= 1 \end{aligned}$$

The journals *Science*, CELL, PNL, and *Nature* are at the top in terms of quality, and also very general for the audience. In the bursting bubbles problem, the author identified that the ratio between surface tension force and gravitational force was large, therefore the effect of bubble bursting must be low. The dimensionless groups that the author found was n , h/R , and $\rho R\gamma/\mu^2$. They illustrate this by showing the graph. The characteristic time and length scale was

$$\begin{aligned} t &= \frac{\tilde{t}}{\omega} \\ x &= \frac{R}{n}\tilde{x} \\ \bar{\sigma}_\times &= 4\mu h \frac{\mathrm{d}u}{\mathrm{d}x} \end{aligned}$$

If there is a high quality paper in such journals, there is often a technical paper that the group releases to show more detail; these are often located in non-top tier journals. In the professor's lab website, the author of the paper, Ortis, walks through the paper. In the term

$$\frac{\omega\mu h^3}{3}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right) - \frac{n^2}{r^2}\right\}^2 f$$

the f gets pulled in as

$$\frac{\omega\mu h^3}{3}\left\{\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}f}{\mathrm{d}r}\right) - \frac{n^2}{r^2}f\right\}^2$$

5 Alternative to Perturbation

2nd order ODE is generally associated with waves. Boundary layer theory:

- ϵ multiplies the highest derivative, and
- the solution breaks down locally, that is, at the boundary layer as $\epsilon \to 0$

The solution can slowly change as it approaches the boundary layer for most of the domain, but rapidly vary in the boundary layer. The local breakdown is due to dissipative forces in the system. The rapidly varying component of the solution decays exponentially away from the local breakdown of the boundary layer.

The solution to the system

$$\begin{aligned} \epsilon y''(x) + y(x) &= 0 \\ y(0) &= 0 \\ y(1) &= 1 \end{aligned}$$

is a fundamentally different problem since there is a global breakdown. Since there are waves for this solution, there is a dispersive behavior. There is a slow change in the wave length and amplitude.

One derivatives such as $y'(x) = y(x)$ are typically exponents, whereas second derivatives $y''(x) = -y(x)$ involves sines and cosines; if there was a $y'(x)$ term in the latter, there would be dampening. If $\exp \mathbb{R}$ would have dissipative effects, where as $\exp \mathbb{I}$ have dispersive.

$\exp(x)$ are growth/decay functions, and sines and cosines are waves.

Midterm will have large problem that covers everything that we did in a very Engineering-ish means.

Local breakdowns are due to dissipation, and leads to boundary layers. Their approximate solution is given as

$$y(x) \approx \sum_{n=0}^{\infty} \epsilon^n y_n(x). \tag{10}$$

The alternative is global breakdown due to dispersion and produces wave-like solution, in which the wavelength is in the order of $\lambda \sim O(\epsilon)$.

WKB Theory: Wentzel Kramers and Brillouin Theory. This can represent a boundary problem or wave-like problem, but only for linear PDE/ODE. The exponent of real numbers produces dissipation, whereas exponent of imaginary numbers produces dispersion. This motivates the approximation:

$$y(x) \approx A(x) \exp\left[\frac{S(x)}{\delta}\right] \tag{11}$$

where $S(x)$ describes a phase. $S(x)$ can vary as a real number, imaginery number, or a constant. If it is real, then we obtain a boundary layer of width δ . If it is imaginary, then we have a region of rapid oscillation, where the wavelength is in the order of $\lambda \sim O(\delta)$. If it is constant, then it will describe the outer solution described by a slowly varying amplitude function.

Both $S(x)$ and $A(x)$ both implicitly depend on δ. We can write

$$y(x) = \exp\left[\underbrace{\frac{\mathbf{i}}{\delta}(S(x) - \underbrace{\mathbf{i}\delta \ln A(x)}_{O(\delta)})}_{\hat{S}(x)}\right] \tag{12}$$

where

$$\hat{S}(x) = \hat{S}_0() + \delta \hat{S}_1(x) + \delta^2 \hat{S}_2(x) + \cdots \tag{13}$$

The approximation for WKB is

$$y(x) \approx \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right],\ s.t. \delta \to 0 \tag{14}$$

Note that ϵ still exists, and may not be equivalent to δ . δ is the wavelength relative to the characteristic scale.

For the midterm, the writeup should appear as a paper. We should show the important steps and talk about the steps taken. We can come to the professor for help.

If there is no dissipitative term, there will not be a boundary layer, typically illustrated as a $u'(x)$ term in the equation. In this case, we can do singular perturbation. When there is a dispersive term, i.e., waves, we have to do WKB approximation, which is only applicable for linear equations:

$$y(x) = \exp\left[\frac{1}{\delta}\sum_{n=0}^{\infty} \delta^n S_n(x)\right] \tag{15}$$

As an example, if we have

$$\begin{aligned} \epsilon y''(x) + y(x) &= 0 \\ y(0) &= 0 \\ y(1) &= 1 \end{aligned}$$

We have to insert the deriatives of equation 15.

6 Multiple Scales

For planetary problems where, for large time scales, deviations become apparent, we can use multiple timescales. We can suppose

$$\begin{aligned} t_1 &= t \\ t_2 &= \epsilon^\gamma t \end{aligned}$$

This converts our ODE to a PDE for a, for example, pendulum equation approximation. The ODE can be converted to the PDE using

$$\frac{\mathrm{d}}{\mathrm{d}t}(\cdot) = \frac{\mathrm{d}t_1}{\mathrm{d}t}\frac{\partial}{\partial t_1}(\cdot) + \frac{\mathrm{d}t_2}{\mathrm{d}t}\frac{\partial}{\partial t_2}(\cdot) = \frac{\partial}{\partial t_1}(\cdot) + \epsilon^\gamma \frac{\partial}{\partial t_2}(\cdot) \tag{16}$$

for the first derivative, and for the second, we get:

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}(\cdot) = \left[\frac{\partial}{\partial t_1}(\cdot) + \epsilon^\gamma \frac{\partial}{\partial t_2(\cdot)}\right]\left[\frac{\partial}{\partial t_1}(\cdot) + \epsilon^\gamma \frac{\partial}{\partial t_2(\cdot)}\right] \tag{17}$$

which is

$$\frac{\partial^2}{\partial t_1^2}(\cdot) + 2\epsilon^{\gamma}\frac{\partial^2}{\partial t_1 \partial t_2}(\cdot) + \epsilon^{2\gamma}\frac{\partial^2}{\partial t_2^2}(\cdot)$$

For the pendulum equation, we would get

$$\theta_0 = c_1(t_2)\cos(t_1 + c_2(t_2)) \tag{18}$$

where the constants are actually functions, and the initial conditions would be

$$c_1(0)\cos(c_2(0)) = 1$$
$$c_1(0)\sin(c_2(0)) = 0$$

This unfortunately tells us only what the value of the function is at the point, not necessarily what the function is.

Mathematician named Noether: under some circumstances, energy is not conserved such as blackholes. Hilbert of Hilbert equation and Klein as in Klein's bottle came up to her with this problem.

Noether's Theorem: If the Langrangian of a system has a continuous symmetry, then there exists an associated quantity which is conserved by the system and vice-versa.

Eulerian Lagragian is about reference frames. In terms of energy though, the Lagrangian or Hamilton is different. Lagrangian says that we want to compute the difference between the kinetic and potential energy:

$$\mathcal{L} = T - V \tag{19}$$

The Lagandra transformation is the conversion of Lagrangian to Hamiltonian. The Hamiltonian is the sum of kinetic and potential energy.

In essence, Lagrangian is one way of computing energy.

Discrete symmetry is present in triangles (at 60 degrees), squares (at 90 degrees), etc.

Continuous symmetry is like a circle in geometry. Continuous symmetry applies to systems. In Physics, changing reference frames is an example of continuous symmetry. The symmetry in the system leads to the conserved quantity. A thrown object has its mass as conserved regardless of reference frame.

In conservation laws, translation in space conserves its linear momentum. In other words, the symmetry of transformation conserves linear momentum. For rotation, angular momentum is conserved. Translation in time yields conservation of energy.

Near a black-hole, space and time are warped.

7 Describing PDEs

Conservation Laws in Mechanics Involve the divergence of some quantity. A faucet is like a source, a drain is like a sink. This is written as $\text{div}\vec{F}(x,y) =$

$\nabla \cdot \vec{F} = F_x' + F_y' + F_z'$ in Cartesian coordinates. This is feasible in other coordinate systems. Divergence of a source is positive, and sink is negative. No change in the vectors is known as 'divergence free'.

Table 1: Divergence Practice

field	variable	static equilibrium
velocity incompressible fluid	$\vec{u}$	divergence free
gravity	$\vec{g}$	negative divergence
magnetic	$\vec{B}$	divergence free
electric	$\vec{E}$	source and sink
heat flux	$\vec{q}$	usually divergence free, but changes in extreme limit
stress	$\bar{\sigma}$	divergence free

Conservation Laws and Divergence Theorem The total volume of sources and sinks is equal to the net flow across the volume's boundary. In 1D, we can do integration by parts to compute easier integrals to do a harder integral: $\int_a^b u\, dv = \Big|_a^b uv - \int_a^b v\, du$. In 2D, this is like measuring the perimeter to calculate the area, i.e., doing a contour integral of the perimeter to compute the area: this is known as 'Green's Theorem'. For an oddly shaped object, we take the flux using the dot product of the vector field and the normal vector: $|\bar{F}| = \vec{F} \cdot \hat{n}$: $\oint_c \bar{F} \cdot \hat{n}\, \mathrm{d}s = \int\int_A \nabla \cdot \vec{F}\, \mathrm{d}A$. In 3D, this is the divergence theorem. In higher dimensions, it is the general Stoke's Theorem.

Conservation of mass can be thought of as continuity. Mass is a strong local form of conservation law. The 'local' part means that it has to be true at all points at any time. The change in amount of stuff per time must be equal to the amount of stuff entering or leaving some area. The latter equality component is the divergence, and can be described as the equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \vec{v} \tag{20}$$

In fluids, the ρ would be fluid density, the $\vec{v}$ would be the fluid momentum (which is related to $\vec{v} = \rho\vec{u}$), and this yields the equation familar in fluids:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\vec{u}) = 0 \tag{21}$$

In thermo, the $\rho = u$ which is the energy density, and the vector field is $\vec{v} = q$ is the energy flux, which yields:

$$\frac{\partial u}{\partial t} + \nabla \cdot \vec{q} = 0 \tag{22}$$

In electromagnetism, we have ρ as the charge density, $\vec{v} = \vec{j}$ is the current density, and this yields

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{j} = 0 \tag{23}$$

Flux is the spatial variation.

∇ is the gradient. The gradient of a scalar is a vector. The gradient of a vector is a tensor. The gradient of a tensor is simply a higher order tensor. A scalar is a rank 0 tensor. A vector is a rank 1 tensor. A tensor is typically a rank 2 tensor, and then there are higher order tensors. The gradient increases the rank.

$\nabla\cdot$ is the divergence. Divergence of a vector is a scalar. The divergence of a tensor is a vector. In other words, the divergence decreases the rank.

$\nabla\times$ is the curl. Curl of a vector is a vector, and curl of a tensor is a tensor. It does not change the rank.

$\nabla^2 = \nabla \cdot \nabla$ is the Laplacian. It yields the same rank since it takes the divergence of a gradient.

Using this, we can define an arbitrary equality $\vec{v} = -c\nabla u$. This allows producing links.

Darcy's Law is fluid flow through a porous material:

$$\vec{q} = -\frac{k}{\mu}\nabla p \tag{24}$$

where ∇p is the pressure gradient, k is the permeability, and μ is the viscosity. In Ohm's law, we have

$$\vec{j} = -\frac{1}{\rho_r}\nabla v \tag{25}$$

where ρ_r is the resistivity, and ∇v is the gradient of the electrical field. In Fick's Law, we have

$$\vec{q} = -D\nabla\phi \tag{26}$$

which is the concentration gradient. In Fourier's law, we have

$$\vec{q} = -k\nabla T \tag{27}$$

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= -\nabla \cdot \vec{v} \\ &= -\nabla \cdot (-c\nabla u) \\ &= c\nabla^2 u \\ \Rightarrow \frac{\partial u}{\partial t} &= c\nabla^2 u \end{aligned}$$

yields the generic heat equation. We could do the conservation of momentum instead, we'd get

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u$$

which is the wave equation.

Finally, we take Laplacians:

$$\nabla^2 u = 0 \qquad \text{equilibrium equation} \tag{28}$$
$$\nabla^2 u = c\dot{u} \qquad \text{heat equation} \tag{29}$$
$$\nabla^2 u = c^2\ddot{u} \qquad \text{wave equation} \tag{30}$$

The Laplacian of a Laplacian is utilized in cases of elasticity since it is easier to solve equations that way. Notice that in the above, on the right, the derivative is w.r.t. to time, which is a scalar.

A linear equation is given as

$$A\partial_x^2 u + B\partial_{xy}^2 u + C\partial_y^2 u + D\partial_x u + E\partial_y u + Fu = G \tag{31}$$

Here, the capital letters can be either constants, or functions of *independent* variables (x and y in this case). This equation is homogenous if $G(x, y) = 0\,\forall(x, y)$.

The type of the equation is important since it aids in understanding how to solve an equation, and whether it can be solved. The basic types of PDEs are

1. Parabolic: $B^2 - 4AC = 0$. For example, $\partial_t u = \partial_x^2 u$ is parabolic. This is generally heat and diffusion equation. Here, information about initial conditions and disturbances are slowly lost.
2. Hyperbolic: $B^2 - 4AC > 0$. For example, $\partial_t^2 u = \partial_x^2 u$ or $\partial_{\alpha\beta}^2 u = 0$ is hyperbolic. This is generally waves and vibrations.
3. Elliptic: $B^2 - 4AC < 0$. For example, $\partial_x^2 u + \partial_y^2 u = 0$ is elliptic. This is generally equilibrium and steady state. Here, any information about initial conditions and disturbances can not be determined.

Something like $y\partial_x^2 u + \partial_y^2 u = 0$ has $B^2 - 4AC = -4y$ changes from elliptic to parabolic to hyperbolic; this can be thought as a shockwave, where in front, in, and behind a shockwave are different.

8 Method of Characteristics

In Method of Characteristics, a PDE is reduced to a family of ODEs. It can be solved using a characteristic curve. For example, in advection/convection with

$$a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

with $u(x, 0) = x_0$ and $u(0, t) = 0$. This equation is like a conveyer belt, where the solution is constant along characteristic curves; this reduces the order: $u(x, t) \to u(x(s), t(s))$ and we'd look for $\frac{\mathrm{d}u}{\mathrm{d}s}$. Here, if the curve is constant, then $\frac{\mathrm{d}u}{\mathrm{d}s} = 0$.

$$\frac{\mathrm{d}u}{\mathrm{d}s} = \frac{\partial u}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial u}{\partial t}\frac{\mathrm{d}t}{\mathrm{d}s}$$

If we assume change of time w.r.t. s to be 1, and x w.r.t. s to be some constant a , we get:

$$\frac{\mathrm{d}u}{\mathrm{d}s} = a\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} = 0$$

with $t(0) = 0$ and $x(0) = x_0$. With this and the above, we get $u(0) = f(x_0)$ and

$$u(x(t), t) = f(x - at)$$

The initial information dictates how any starting point continues.

9 Separation of Variables

Guessing and checking can be used to solve the equation. Here we

1. Assume the solution is a product of independent variables of two independent variables.
2. Collect the functions of same variable on the same side of the equation
3. Assume both equations are equal to a constant
4. Solve separately and recombine

Laplace's equation in 2D:

$$\nabla^2 u(x, y) = 0$$

Here, we say that the ansantz $u(x, y) = X(x)Y(y)$. Then

$$\frac{\partial^2}{\partial x^2}(XY) + \frac{\partial^2}{\partial y^2}(XY) = 0$$

Since X is not function of y and Y is not function of x , we get the ordinary derivatives

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2}Y + \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2}X = 0$$

We separate the variables and assume they are equal to a constant

$$\frac{X''}{X} = \frac{Y''}{Y} = \lambda \Rightarrow \begin{cases} X'' - \lambda X &= 0 \\ Y'' + \lambda Y &= 0 \end{cases}$$

We can consider different λ. If $\lambda > 0$, we'd get

$$X(x) = c_1 \exp\left(x\sqrt{\lambda}\right) + c_2 \exp\left(-x\sqrt{\lambda}\right)$$
$$Y(y) = c_3 \cos\left(y\sqrt{\lambda}\right) + c_4 \sin\left(y\sqrt{\lambda}\right)$$

or if we get $\lambda = 0$

$$X(x) = c_1 x + c_2$$
$$Y(y) = c_3 y + c_4$$

or if we get $\lambda < 0$

$$
\begin{aligned}
X(x) &= c_1 \cos\left(x\sqrt{\lambda}\right) + c_2 \sin\left(x\sqrt{\lambda}\right) \\
Y(y) &= c_3 \exp\left(y\sqrt{\lambda}\right) + c_4 \exp\left(-y\sqrt{\lambda}\right)
\end{aligned}
$$

Notice that the terms with exponents would yield exponential growth or decay, whereas the sine and cosine would yield wave-like solution.

Then, combine the two to get $u(x,y) = X(x)Y(y)$.

As an example, a vibrating guitar string has the dynamics

$$\partial_t^2 u = c^2 \partial_x^2 u$$

with the initial conditions $u(x,0) = f(x)$ and $\partial_t u(x,0) = 0$, and the boundary conditions $u(0,t) = 0 = u(L,t)$. These are valid for $0 < t < \infty$ and $0 < x < L$. Note that $c = \sqrt{T/l}$.

Now, we assume that we know the solution to be

$$u(x,t) = X(x)T(t).$$

A proper string has 0 resistance to bending, but a beam does. A vibrating guitar string actually behaves like a beam.

A separation of variables that yields eigenvalues and eigenfunctions are called Sturm-Liouville problem. These pop out when there are IC/BC that contain a derivative.

In a 1D heat flow with insulation along the length, with cooling at one end.

$$\partial_t u = \alpha^2 \partial_x^2 u$$

with IC $u(0,t) = 0$ and BC $u(x,0) = x$ and $\partial_x u(1,t) + hu(1,t) = 0$. Doing separation of variables $u(x,t) = X(x)T(t)$, we get

$$\frac{T'}{\alpha^2 T} = \frac{X''}{X} = \pm\lambda^2$$

Here, the sign of λ must be negative since we need wave like decay in temperature. If it was positive, we would see temperature blowing up.

$$
\begin{aligned}
T' + \lambda^2\alpha^2 T &= 0 \\
\Rightarrow T(t) &= A\exp(-(\lambda\alpha)^2 t) \\
X'' + \lambda^2 X &= 0 \\
\Rightarrow X(x) &= B\sin(\lambda x) + C\cos(\lambda x) \\
\Rightarrow u(x,t) &= e^{-(\lambda\alpha)^2 t}\left(D\sin(\lambda x) + C\cos(\lambda x)\right)
\end{aligned}
$$

The roots of a solution are the eigenvalue, which can be used to find the eigenfunction. Before applying initial conditions, if there are multiple solutions, such as

$$u(x,t) = \exp(-(\beta_n\alpha)^2 t)\sin(\beta_n x)$$

then it can be written as

$$u(x,t) = \sum_{n=1}^{\infty} a_n X_n(x) T_n(x)$$

if the initial condition is $u(x,0) = x$, then we get

$$u(x,0) = x = \sum_{n=1}^{\infty} a_n \sin(\beta_n x)$$

The summation component is difficult to remove. However, since this is a trigonometric function, we can take advantage of orthogonality. We multiply both sides with $\int_0^1 \sin(\beta_m \tilde{x})\, d\tilde{x}$:

$$\int_0^1 \tilde{x} \sin(\beta_m \tilde{x})\, d\tilde{x} = \sum_{n=1}^{\infty} a_n \int_0^1 \sin(\beta_n \tilde{x}) \sin(\beta_m \tilde{x})\, d\tilde{x}$$

where, if $m = n$, then the right hand side is a finite number, and the left hand side is also a finite number. This allows us to solve for a_n as

$$a_n = \frac{2\beta_n}{\beta_n - \sin\beta_n \cos\beta_m} \int_0^1 \tilde{x} \sin(\beta_n \tilde{x})\, d\tilde{x}$$

10 Transforming PDEs

In general, the primary tactic in solving PDEs is transforming hard PDEs into easier ones. Galilean transform, Lorentz transform, and Hankel transform involve transforming the coordinates. Laplace transform involves the independent term, say, space to frequency. One of the most powerful ones is the Fourier transform. A Legendre transform is a common transform that is intuitively done.

The Galilean transform is one of the transformations that AI currently is not good at. These transformations can be considered to be a 'Separation of Physics'.

If we consider a rod with some initial temperature distribution that is smooth, and can consider boundary conditions at the end: $\phi(x) = u(x,0)$ and $u(0,t) = u(1,t) = 0$. Suppose the temperature leaves the bar, therefore, it will diffuse along the bar, and over time also decrease.

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} - \beta u \tag{32}$$

We can consider that the effects are happening separately to solve this. Two phenomena are diffusion $\alpha^2 \partial_x^2 u$ and heat flow across boundary $-\beta u$. We can turn off some Physics by changing the constant. Suppose $\alpha = 0$ which yields

$$\frac{\partial u}{\partial t} = -\beta u \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}t} - \beta u$$

which is an ODE that has a solution $u_{HF}(x,t) = c_1 e^{-\beta t}$. We can now suppose that the solution is $u = e^{-\beta t} \underbrace{w(x,t)}_{\text{diffusion}}$:

$$\begin{aligned}
\frac{\partial w}{\partial t} &= \alpha^2 \frac{\partial^2 w}{\partial x^2} \\
w(0,t) &= w(1,t) = 0 \\
w(x,0) &= \phi(x) \\
\text{separation of variables: } w(x,t) &= \sum_{n=1}^{\infty} a_n e^{-(n\pi\alpha^2)t} \sin(n\pi x) \\
a_n &= 2 \int_0^1 \phi(\bar{x}) \sin(n\pi\bar{x})\, d\bar{x}
\end{aligned}$$

Note that, although there are two decaying exponentials, we do not merge these since it describes different phenomena (insulation vs diffusion).

The diffusion-convection equation can be written as

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} - v \frac{\partial u}{\partial x} \tag{33}$$

where the diffusion component can be seen as a wave diffusing to its surroundings, whereas convection is a wave moving. Here, v is a constant velocity, and u_x is the gradient of the moving medium.

11 Integral Transform

We take a hard problem with function $f(x)$ and transform it into a new easier problem $F(s)$ that is easy to solve, and then we do inverse transform to solve it. In general:

$$F(s) = \int_A^B K(s,x) f(x)\, dx \tag{34}$$

where $K(s,x)$ is the kernel of the integral transform.

An integral transform eliminates a partial derivative with respect to one of the variables: $u(x,t) \to U(s)$. In this case, we got an ODE from a PDE. This resolves a function into a spectrum of its components. Doing this on a sum of sine functions would reveal the different frequencies (such a transform showcases the Fourier series). For example, if

$$f(x) = \sum_{n=0}^{\infty} [A_n \cos(nx) + B_n \sin(nx)]$$

where our constants A_n and B_n describe 'how much something is made of'. The spectrum of $f(x)$ would be

$$C(f) = \sqrt{A_n^2 + B_n^2}. \tag{35}$$

Worked example:

$$f(x) = 1 + \sin x + \frac{1}{5}\sin 3x + \cos x + \frac{1}{2}\cos 2x + \frac{1}{4}\cos 4x$$

where the table 2 showcases the spectrum. Periodic functions have discrete spectrums.

Table 2: $C(f)$ table

n	A_n	B_n	$C(f)$
0	1	0	1
1	1	1	$\sqrt{2}$
2	0.5	0	0.5
3	0	0.2	0.2
4	0.25	0	0.25

11.1 Fourier Series

An a-periodic function, on the other hand, has a continuous spectrum:

$$f(x) = \int_{-\infty}^{\infty} k(\omega)\cos(\omega x) + j(\omega)\sin(\omega x)\, dx \tag{36}$$

This still has a frequency spectrum, but instead is continuous

$$C(f) = \sqrt{k^2(\omega) + j^2(\omega)} \tag{37}$$

The frequency spectrum is useful as a fingerprint to analyze things, such as Chemistry, or Shazam using fast fourier transform, or the Marlin bird app.

A Fourier series is valid when there is a periodic function defined on an infinite domain $(-\infty, \infty)$. Earlier in the course, the 1D diffusion equation was solved using similarity variable, but can also be solved using the Fourier series. A periodic function defined on a finite interval is also valid.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \tag{38}$$

where the sines and cosines are the harmonics, and a_n and b_n are the Fourier coefficients. This was originally developed to study heat transfer problem. The Fourier series is capable of modelling periodic functions with sharp edges if we take an infinite terms.

We can obtain the terms a_n and b_n by taking advantage of orthogonality:

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos(x)\left(\frac{n\pi x}{L}\right) dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(x)\left(\frac{n\pi x}{L}\right) dx$$

which all can be computed since these are simply numbers. For the sawtooth function, our $a_n = 0$, and $b_n = -\left(\frac{2L}{n\pi}\right)(-1)^n$.

11.2 Fourier Integral

One of the limitations of this transform is non-periodic function on an infinite domain. The analogous representation here is the Fourier integral.

$$f(x) = \int_0^\infty a(s)\cos(sx)\,ds + \int_0^\infty b(s)\sin(sx)\,ds \tag{39}$$

where

$$\begin{aligned} a(s) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(sx)\,dx \\ b(s) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin(sx)\,dx \\ C(s) &= \sqrt{a(s)^2 + b(s)^2} \end{aligned}$$

In the earlier saw tooth function, a lot of high frequencies are needed to model the sharp edges. In compression algorithms, the frequencies that are not really visible or useful are removed.

11.3 Fourier Transform

Euler's relation is the beautiful connection

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{40}$$

which allows us to perform

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx\right]e^{isx}\,ds \tag{41}$$

is the Fourier Integral representation, which contains both the Fourier Transform and the Inverse Fourier Transform.

$$\mathcal{F}[f] \equiv F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx \tag{42}$$

is the Fourier transform of f, where the kernel is $K(s,x) = e^{-isx}$. Notice that when we do this integral, the x disappears, and s appears instead. The choice of the sign term in the kernel is dependent on the tool used: Mathematica uses a positive sign. Whichever software is used, we should ensure that the sign is correct, and ideally it should be negative; in Mathematica, we have to specify it to be negative.

Similarly, the inverse of the Fourier transform is

$$\mathcal{F}^{-1}[F] = f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}\,ds \tag{43}$$

A couple of notes on Fourier transform:

- Not all functions of a FT. The integral may not exist (divergence might happen, such as x which goes to infinity)
- Functions that decay or go to 0 sufficiently fast as $|x| \to \infty$ have a FT
- The $|\mathcal{F}[f]|$ is the frequency spectrum
- The FT of $f(x)$ may be complex, such as $f(x) = e^{-x}$ for $0 < x < \infty$, which is $F(s) = \frac{1}{\sqrt{2\pi}}(\frac{1-is}{1+is^2})$.
- The FT turns the operation of differentiation into multiplication.
- The FT is linear: $\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f] + b\mathcal{F}[g]$

Notice that, since the kernel has a complex number, it creates waves which helps pick out frequencies. We use the Fourier transform to go from differentiation to multiplicative which is easier, do our math, and then perform inverse Fourier.

When transsforming partial derivatives, we have to consider what we are transforming. If we, say, are transforming partial derivatives w.r.t. x , then the FT would include dx; however, this transform would be different than taking the transform w.r.t. other variables, say y. For example:

$$\begin{aligned}
\mathcal{F}[\partial_x u] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial x} e^{-isx}\, dx \\
&= is\mathcal{F}[u] \\
\mathcal{F}^{[}\,[\partial] - x^2 u] &= -s^2 \mathcal{F}^{[}\,[u]] \\
\mathcal{F}^{[}\,[\partial]_t\, u] &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-isx}\, dx \\
&= \frac{\partial}{\partial t} \mathcal{F}^{[}\,[u]] \\
\mathcal{F}^{[}\,[\partial]_t^2\, u] &= \frac{\partial^2}{\partial t^2} \mathcal{F}^{[}\,[u]]
\end{aligned}$$

The FT is linear, so a powerful thing to recognize is that

$$\mathcal{F}^{[}\,[a]\, f(x) + bg(x)] = a\mathcal{F}^{[}\,[f]] + b\mathcal{F}^{[}\,[g]]$$

and this is useful since we will often find something like $\mathcal{F}^{[}\,[f]]\mathcal{F}^{[}\,[g]]$ that we want to take the inverse of, but note that

$$\mathcal{F}^{[}\,[f]\, g] \neq \mathcal{F}^{[}\,[f]]\mathcal{F}^{[}\,[g]].$$

Instead, we think of the convolution:

$$\mathcal{F}^{[}\,[f] * g] = \mathcal{F}^{[}\,[f]]\mathcal{F}^{[}\,[g]] \tag{44}$$

which is important to note since $f * g = \mathcal{F}^{-1}\left[\mathcal{F}^{[}\,[f]]\mathcal{F}^{[}\,[g]]\right]$, which is something we can compute to undo the convolution since

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x - s)g(s)\, ds \tag{45}$$

In 1D diffusion for

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$$
$$u(x,0) = \Phi(x)$$
$$x \in (-\infty, \infty)$$

Taking the fourier transform, we obtain

$$\frac{\partial}{\partial t} U + \alpha^2 s^2 U = 0$$

which has a solution of

$$U(t) = c_1(s) \exp(-s^2 t \alpha^2) = \Phi(s) \exp(-s^2 t \alpha^2).$$

and we take the inverse transform of this, while knowing that the result involves a convolution:

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(s) \exp\left(-\frac{(x-s)^2}{4\alpha^2 t}\right) ds$$

11.4 Hankel Transform

Hankel transform can be considered a transformation for 2D. A rock falling on a still pool of water and creating ripples is a cylindrical/spherical problem; the periodic waves are described by Bessel functions, which works when there is an axis of symmetry.

Bessel functions of the first kind are denoted with $J_\alpha(r)$. In the first kind, $J_0(0)$ has some non-zero finite value, $J_1(0) = 0$, and $J_2(0) = J_2'(0) = 0$. In the second kind, there is a singularity at the origin, where in $Y_\alpha(r)$, the α simply showcases how spread the singularity is. Bessel functions appear as solutions to Laplace's equation $\nabla^2 u = 0$ and the Helmholtz equation $\nabla^2 u = k^2 u$, when these equations are projected in cylindrical/spherical coordinates, assuming that there is an axis of symmetry; values can vary only in the radial direction. Note that, this is not valid for something like waves on a sphere, these would be described by Lajandra polynomials.

If we have an equation like

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + (r^2 - \alpha^2) u = 0$$

The Hankel function looks like

$$H_\alpha^{(1)}(r) = J_\alpha(r) + iY_\alpha(r) \tag{46}$$
$$H_\alpha^{(2)}(r) = J_\alpha(r) + iY_\alpha(r) \tag{47}$$

and the Hankel transform is

$$\mathcal{H}_\alpha[f] = \int_0^\infty f(r) J_\alpha(kr) r \, dr \equiv F_\alpha(k) \tag{48}$$

$$\mathcal{H}_\alpha^{-1}[f] = \int_0^\infty F_\alpha(k) J_\alpha(kr) k \, dk \equiv f(r) \tag{49}$$

For Bessel functions, one kind does not have singularities in the center, the other does. The kernel in the Hankel transformation is the Bessel function. Note that we also have to integrate with r or k since we have an axis of symmetry. There are bounds on α such that $\alpha \geq \frac{1}{2}$, which can take fractional or integer. If α is an integer, then it would be for cylindrical coordinates. Half integers are for spherical coordinates. Laplace transform works on differentiation, whereas Hankel works on differential *operators*.

In general, where our differential operator has a θ component,

$$\mathcal{H}_\alpha \left\{ \left(\nabla^2 - \frac{n^2}{r^2} \right) f(r) \right\} = \mathcal{H}_\alpha \left\{ \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}f}{\mathrm{d}r} \right) - \frac{n^2}{r^2} f(r) \right\} = -k^2 F_\alpha(k) \tag{50}$$

In particular:

$$\mathcal{H}_0 \left\{ \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}f}{\mathrm{d}r} \right) \right\} = -k^2 F_0(k)$$

and

$$\mathcal{H}_1 \left\{ \frac{1}{r} \frac{\mathrm{d}}{\mathrm{d}r} \left(r \frac{\mathrm{d}f}{\mathrm{d}r} \right) - \frac{1}{r^2} f(r) \right\} = -k^2 F_1(k)$$

11.5 Laplace Transform

The variable has to be from $0 < t < \infty$, though not necessarily time. ODEs get transformed to algebraic equations. It is given as

$$\mathcal{L}[f] = \int_0^\infty f(t) e^{-st} \, dt = F(s) \tag{51}$$

and its inverse is given as

$$\mathcal{L}^{-1}[F] \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) e^{st} \, dt = f(s). \tag{52}$$

As an example

$$\mathcal{L}\left[\frac{\partial u}{\partial t}\right] \int_0^\infty \frac{\partial u}{\partial t} e^{-st} \, dt = sU(x,s) - u(x,0)$$

for the first derivative, and for the second derivative,

$$\mathcal{L}\left[\frac{\partial^2 u}{\partial t^2}\right] = \int_0^\infty \frac{\partial^2 u}{\partial t^2} e^{-st} \, dt = s^2 U(x,s) - su(x,0) - \frac{\partial u}{\partial t}(x,0)$$

The Convolution looks like

$$(f * g)(t) = \int_0^t f(\tau)g(t-\tau)\,d\tau = \int_0^\tau f(t-\tau)g(\tau)\,d\tau$$

which is a finite convolution.

$$\mathcal{L}\left[\frac{\partial u}{\partial x}\right] = \frac{\mathrm{d}U}{\mathrm{d}x}$$

12 Variational Calculus

All the earlier transforms are local; that is, we consider the meaning of divergence, of sources and sinks. Instead of doing force balance at every single local point, we will now consider global. Global conversation laws is like saying energy is conserved in the system. A local law is like $\dot{\rho} = -\nabla\vec{v}$, whereas a global law is of the type $\mathcal{L} = \int(\cdot)\,\mathrm{d}V$, which we would either minimize or maximize $\mathcal{L}$. This is the priciple of minimum potential energy and principle of least action.

Extremizing is about finding max or min. The derivative of a function $f(x)$ yields zero when there are minimums and maximums. Taking the second deravitive enables us to determine if it is a minimum or maximum.

In variational calculus, we no longer work with simple functions. We deal with *functionals*, which is function of a function of variable. Operation on the function of a variable is ordinary calculus; and the operations on function of a function of a variable is variational calculus.

The energetic cost of bending a beam would be

$$\mathcal{U}_b = \frac{B}{2}\int_0^L \kappa^2(s)\,ds$$

Here, $\mathcal{U}_b$ is the strain energy and is a function of κ which is a function of s . The dimension is same as the number of variables. Control theory often utilizes this because they care about minimization of certain constraints.

The $\mathcal{L}$ is the Langrangian

$$\mathcal{L} = T - V \tag{53}$$

In some problems, it is convenient to use the Hamiltonian instead

$$\mathcal{H} = T + V \tag{54}$$

and the Lajendre transfrom can convernt between the two.

We'd maximize something like energy energy, whereas we would maximize entropy or geometry. This involves finding a line, surface, or valume. We would perturb the system to find the governing differential equation to directly numerically compute the solution. COMSOL and other solvers use variational computations, even if we insert the PDEs; since it interpolates, there are errors at every point, and therefore tries to obtain an average etc; also called weak form. We would generate a perturbed solution $\bar{x}(t)$ which should be equal to

$x(t) + \epsilon + \eta(t)$, where $\eta(t)$ is any arbitrary function, and $x(t)$ is the unknown solution. This is more typically written as

$$\bar{x}(t) = x(t) + \delta x(t) \tag{55}$$

where δ indications the first variation. $\delta^2 x$ is the second variation. This indicates a tiny infinitisimal change.

When we perturb the state variables, it is typcially $\mathcal{L} \to \vec{u}$ and $\dot{u}$.

Procedure:

1. Let's say our functional is $F(x;\, u, u')$ where u, u' are dependent variables.
2. We pertrub the dependent variables by adding small changes $F(x;\, u + \delta u, u' + \delta u'$. The presence of the derivative term is a problem.
3. Perform the Taylor series expansion
$$F(x; u+\delta u, u'+\delta u') = F(x; u, u') + \frac{\partial F}{\partial u}\delta u + \frac{\partial^2 F}{\partial u^2}\delta^2 u + \frac{\partial F}{\partial u'}\delta u' + \frac{\partial^2 F}{\partial u'^2}\delta^2 u' + 2\frac{\partial^2 F}{\partial u \partial u'}\delta u \delta u' + \mathcal{O}(\cdots)$$
4. Compare 'solution' to perturbed solution:
$$F(x; u + \delta u, u' + \delta u') - F(x; u, u') = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u' = \delta F$$
This is an important quanitity since if we find the min or maximum, the quantity $\delta F = 0$. We can get rid of the $\delta u'$ term using integration by parts. The first variation being equal to 0 yields the static equilibrium, because this states that the derivative of energy with respect to a distance is a force which has to be at an equilibrium. The second variaton tells us about stability. $\partial F = 0$ is equilibrium, and $\partial^2 F > 0$ indicates stability.

For example, in

$$\mathcal{L} = T - V$$

where T is kinetic energy and V is some potential energy, we get

$$\mathcal{L}(t, x, x') = \frac{m}{2}(x')^2 - V(x)$$
$$\delta\mathcal{L} = \frac{\mathcal{L}}{x}\delta x - \frac{\delta L}{\delta x'}\delta x'$$
$$\delta\mathcal{L} = m x' \delta x' - V'(x)\delta_x = 0$$

The δx can be an arbitrary non-zero perturbation whereas $\delta x'$, so we'll try to remove it.

Principle of least action

$$\delta S = \int_{t_1}^{t_2} \delta\mathcal{L}\, dt \tag{56}$$

We perform integration by parts

$$\int_{t_1}^{t_2} mx'\delta x'\,dt = \underbrace{mx'\delta x\Big|_{t_1}^{t_2}}_{\text{describes bcs}} - \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(mx')\delta x\,dt$$

We get

$$\delta S = \int_{t_1}^{t_2} (-mx'' - V'(x))\delta x\,dt = 0 \tag{57}$$

Since $\delta x \neq 0$, and the integral has to yield 0, we are required to set

$$mx'' + V'(x) = 0 \Rightarrow F = ma \tag{58}$$

Suppose there is a beam fixed at one end, and on the other end there is a force going in P. There is also a constant load throughout the beam. It has a specified length L. We get

$$\mathcal{V}(x; u, u') = \int_0^L \left[\frac{EA}{2}\underbrace{(u')^2}_{\text{strain}} - \underbrace{qu}_{\text{distributed}}\right] dx\; \underbrace{-Pu(L)}_{\text{external load}}$$

Here, we compute $\delta\mathcal{V}$, solve for $\delta V = (\cdot)\delta u$ using integration by parts, solve for the governing equation, and then set $q = 0$.

The equation $\sum \vec{F} = m\vec{a}$ and $\delta\mathcal{L} = 0$ are identical, the latter being the 'weak form'. The latter is useful for cases where all forces are internal and geometry is changing. Red blood cell is an example. Red blood cells have constant volume and surface area, but can change shape. Variational methods allows us to model this. For this case, $\mathcal{L} = -V = -(U_{\text{bending}} + \text{constraints (area, vol)})$. Here, the variation is

$$\mathcal{V} = \int_a^b \mathcal{L}(x; y, y')\,dx$$

and

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial y}\delta y + \frac{\partial\mathcal{L}}{\partial y'}\delta y' = \frac{\partial\mathcal{L}}{\partial y}\delta y - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial\mathcal{L}}{\partial y'})\delta y = \left[\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial\mathcal{L}}{\partial y'})\right]\delta y = 0$$

Here, we had to take integration by parts. Additionally,

$$\frac{\partial\mathcal{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}(\frac{\partial\mathcal{L}}{\partial y'}) = 0$$

This is the Euler-Langrange equation, and when doing the integration by parts earlier, the natural boundary condition is

$$\frac{\partial\mathcal{L}}{\partial y'}\delta y\Big|_a^b = 0$$

To minimize a functional subject to some constraint, we utilize a Lagrange Multiplier, typically given the symbol λ. This always increases the number of equations that have to be solved. It can be solved analytically.

The alternative to the Langrange Multiplier is the Penalty Function method. No new unknowns are added, but we do add a term that 'penalizes' current states. If we consider energy as the currency of the universe, the universe will conserve as much energy as possible. However, this means adding a large arbitrary number to our equation so that a certain solution never shows up. However, this means that we are satisfying the constraint only approximately. This is commonly used analytically. Note that the penalty is actually not infinite.

$$\mathcal{V}(x; u, u', v, v') = \int_a^b \mathcal{L}(x; u, u', v, v' \, dx$$

Note that we set the constraint to be 0 and find λ in $\lambda(K_{def} - K) = 0$. So here, we will call our constraint $G(x; u, u', v, v') = 0$. We will add $\int_\Omega \lambda G \, \mathrm{d}\Omega$ to the energy functional:

$$\mathcal{V} \int_a^b (\mathcal{L} + \lambda G) \, \mathrm{d}x$$

Now if we take the variation:

$$\delta\mathcal{V} = \int_a^b (\delta\mathcal{L} + (\delta\lambda)G + \lambda(\delta G)) \, \mathrm{d}x = 0$$

notice how we had to do kind of a chain rule on λG.

In variational calculus, we establish equilibrium by setting $\delta\mathcal{V} = 0$; variational tools also enables understanding stability, which is determined by $\delta^2\mathcal{V} > 0$. The latter requires the second variation to be positive-definitive. This is like the second-derivative. When we take the first differential, we perturb again, to see how stable that state is. The stability can be stable equilibrium, unstable equilibrium, and neutral equilibrium.

This can get complicated if it is in a local low energy state; these are called meta-stable state because small perturbation keep it stable, but larger perturbations can make it unstable. Similarly, something can be meta-unstable. We could perturb it in many directions depending on dimensionality of independent variables.

We often end up checking the third differential $\delta^3\mathcal{V} > 0$.

13 Stability via Graphical Methods

Professor highly recommends Stephen Strogratz's Non Linear Dynamics and Chaos book.

For example, $\frac{\mathrm{d}x}{\mathrm{d}t} = \sin(x(t)$ with $x(0) = x_0$. The analytical closed form solution for this is

$$t = \ln \left| \frac{\csc(x_0) + \cot(x_0)}{\csc(x) + \cot(x)} \right|$$

We convert to phase space potraits, where we define $x_1 \equiv x$ and $x_2 = \dot{x}$. We draw a phase potrait of how the slope varies with the position. The points where the graph crosses the x_1 axis are the equilibrium states.

Mathematica has a `StreamPlot[]` function for phase plots such as this. For the pendulum with $\ddot{x} + \lambda \sin x = 0$, we set $x_1 = x$ and $x_2 = \dot{x}$, where $\dot{x}_2 = -\lambda \sin x_1$.

A phase plot for spring-mass system $m\ddot{x} + a\dot{x} + kx = 0$ looks similar.

Stability of a system can change based on the control parameters. Fixed points of stability are equilibrium solutions, typically denoted by x^*. In $\dot{x} = x^2 - 1$, there are two points of equilibrium: -1 and 1. However, only the former is stable since it brings it back to the original location.

Resource-constrained population growth is given by $\dot{N} = rN(K-N)$, where K is the carrying capacity.

Linear stability analysis by $x(t) = x^* + \eta(t)$ where $\eta(t)$ is the small perturbation.

$$\dot{\eta} = \frac{\mathrm{d}}{\mathrm{d}t}(x(t) - x^*) = \dot{x} = f(x) \tag{59}$$

Here,

$$f(x) = f(\underbrace{x^*}_{\text{equil.}} + \underbrace{\eta(t)}_{\text{perturb.}}) = \cancelto{0}{f(x^*)} + \eta f'(x^*) + \cancelto{0}{\mathcal{O}(\eta^2)} \Rightarrow \dot{\eta} = f(x^*) = \eta f'(x^*) \tag{60}$$

This states that either $\dot{\eta} = \eta = e^t$ for case where $f'(x^*) > 0$, or $\dot{\eta} = -\eta = e^{-t}$ for $f'(x^*) < 0$.

In a Saddle Node, such as a Josephson Junction, that has an insulator between super conductors. Used in Electrical Engineering, and the governing equation for this system is

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}t^2} + b\frac{\mathrm{d}\phi}{\mathrm{d}t}\sin\phi = I \tag{61}$$

and is a lot like the pendulum equation. The second order derivative is very small, so it looks more like an over-damped pendulum. We would non-dimensionalize this after removing the highest order derivative. We get the key term

$$\dot{\theta} = \gamma - \sin(\theta) \tag{62}$$

which does not look like the bifurcation equation (which involves $\dot{\theta}^2$ instead of $sin\theta$). When $\gamma \leq 1$, two fixed points appear. This is equivalent to a pendulum in honey; at a certain point, the pendulum would be at an angle but still not move. Note that for $\gamma > 1$, it will keep moving. When there's two, the other is the stable equilibrium.

Transcritical Bifurcation has the form

$$\dot{x} = rx - x^2 \tag{63}$$

and this always has a fixed point. This can be seen in population growth

$$\dot{N} = rN(1 - N/k) \Rightarrow \dot{x} = rx - \frac{r}{k}x^2$$

One of the fixed point is always at $x = 0$, and other is dependent on r. Here, fixed points undergo a change of stability:

- $r > 0$, stable f.p. at r, unstable at 0
- $r = 0$, unstable on left of 0, and stable to the right.
- $r < 0$, stable at f.p. at 0 , and unstable at r

Behavior of a solid state laser. When we pump energy into the active material, it gets excited. We excite in one direction, there are mirrors in perpendicular sides. At one point, this goes from being a lamp to a laser. At some critical point, all neighboring points emit same wavelength. Here, the equation is $\dot{n} = rn$ where r defines how much energy goes into exciting the atoms vs being absorbed by surrounding, hence this involves a gain and loss term. The loss term is $k = 1/\tau$ where τ is the timescale of the lifestyle of the photon. The gain term is given as GN where $N(t)$ is the number of protons in excited state. This yields

$$\dot{n} = (GN(t) - k)n$$

We can re-write $N(t) = N_0 - \alpha n$ where N_0 total number of atoms we could have, and αn is the number of photons that are released, and α is the rate at which atoms return to ground state. This yields the more familar equation

$$\dot{n} = (GN_0 - k)n - (\alpha G)n^2 \tag{64}$$

where the critical point happens at $N_0 = \frac{k}{G}$.

In pitchfork bifurcation, an instability happens that splits of paths. Common in physical problems with similarity, such as Euler buckling. Fixed points appears and disappears in pairs. In contrast, transcritical fixed points always exist but change stability. For pitchfork, as the load on a vertical bar becomes $P > P_c$, it could buckle left or right. There are two types.

In supercritical, stable fixed points are generated after bifurcation. Euler buckling is supercritical because the buckled mode is stable. The equation for this is

$$\dot{x} = rx - x^3 \tag{65}$$

Here, for $r > r_c$, two stable fixed points emerge, where far left and right are stable regions, and third unstable fixed point is unstable. For $r < r_c$, only an unstable fixed point exist at $x = 0$.

Subcritical is the other type, and this is concerning because this generates unstable fixed points. These systems sare incredibly sensitive to imperfections. These are stabilized by higher-order terms. These factors lead to catastrophic instabilities. A beam slowly bending is a slow failure, whereas catastrophic is a soda can being crushed immediately. Equation for this is

$$\dot{x} = rx + x^3 \tag{66}$$

This has a diagram that is mirrored about the horizontal axis. This can be stabilized if we had a negative higher order term, such as $-x^5$ This causes sudden jump.

Supercritical smoothly transitions to new equilibrium state, whereas subcritical suddenly jumps to new equilibrium state.

$\dot{x} = f(x) = -\frac{\mathrm{d}V}{\mathrm{d}x}$ is the negative gradient of the potential energy. At equilibrium, this is zero. We have to see the second derivative for stability; stable if positive.

For a simple buckling with a beam vertical on a pivot that has a spring, the energy is $u_H = 0.5 k_r \theta^2$ and work done is $W = Pu = PL(1 - \cos\theta)$