ENGME 519
Theory of Heat Transfer

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Book used in this course is 'A Heat Transfer Textbook' by John Lienhard V and IV, Sixth Edition.

A second order equation can be reduced into first order (reduction of order):

$$\begin{aligned} f''(x) + p(x)f'(x) &= c \\ \Rightarrow g'(x) + p(x)g(x) &= c \\ \text{where } g(x) &= f'(x) \end{aligned}$$

To solve first order differential equation, if we have

$$g'(x) + p(x) \cdot g(x) = c$$

we set $\mu(x) = e^{\int p(x)\,\mathrm{d}x}$. Multiplying both sides with $\mu(x)$, you'd notice that

$$\mu(x)p(x) = \frac{\mathrm{d}\mu(x)}{\mathrm{d}x}.$$

Substituting this, we realize that product rule of derivatives apply, and we can simply integrate:

$$\begin{aligned} \mu g' + \mu' g &= \mu c \\ \Rightarrow \frac{\mathrm{d}}{\mathrm{d}x}(\mu g) &= \mu c \\ \Rightarrow \mu g &= \int \mu c \,\mathrm{d}x \end{aligned}$$

and we simply solve for g next. Since

$$f'(x) = g(x)$$

we simply take another integral now. Do not forget constants!

1 Lecture 1

We should read chapter 1 as of now, and then chapter 2. Chapter 3 won't be read. No information about grades because the priority is learning, not the grades. Lot of homework. Two tests, one in the middle, and one in the end. We can also do a project if we want. The first test will be before the spring break. The project would supplement the grade.

We start with a body that has a continuous distribution of matter (in contrast to atomistic). We'll assume that the material body is smooth. Furthermore, we won't get into molecular models in this course. This enables us to use calculus. We'll have radiation near the end, however, despite not having matter, radiation does have continuous space. At each point of a body, there's a temperature distribution, $T(x, y, z, t)$, which is called a temperature field. If temperature differences exist, then over time, the temperature will even out; inside the body, there is a flow of energy from hot to cold, called the heat flux vector $\vec{q}(x, y, z, t)$. The dimensions of $\vec{q}$ are energy per unit area per unit time. Flux is power per unit area.

If we have a surface on a body with a normal vector and a heat flux vector, $\vec{q}$, we can find the flow in a particular direction: $\mathrm{d}Q =$ energy per time through $\mathrm{d}A = (\vec{q} \cdot n)\,\mathrm{d}A$. This is similar to a fluid, and it used to be considered a fluid. Although no matter is flowing, we still use fluid flow terminology. Throughout this course, we will attempt to describe the flow of heat through something. We want to understand how heat flows in a body, design systems that make heat flow in a certain way, or prevent it.

Three types of heat transfer exist. Conduction is the heat transfer in a stationary solid. Convection is heat transfer in a moving fluid, which can be a liquid or a gas. There is a gray area where heat flow can transition from conduction to convection, however, this can primarily be considered convection. Radiation is the heat transfer in empty space, via electromagnetic waves.

First physical principle is the relation between the temperature and the heat flux vector. The material body can be fluid or solid. Fourier's law of conduction is present here. In a continuous distribution of matter, the temperature $T(x, y, z, t)$ which is a scalar field, and the heat flux vector which is a vector field $\vec{q}(x, y, z, t)$ are related by the Fourier's Law of Conduction:

$$\vec{q} = -k\nabla T. \tag{1}$$

It says that if we know the temperature distribution, and the thermal conductivity k of the material, then heat will flow if there is a gradient in temperature as a function of position. The negative sign ensures that the heat travels from hot to cold.

An isothermal surface is a surface in a body that has the identical temperature. If we have three isotherms $T_1 > T_2 > T_3$, then the gradient will point from T_3 to T_1, and be perpendicular to the isotherm; the heat flow is negative, hence heat flows T_1 to T_3.

If we assumed a 2D object instead, we'd use isothermal curves instead of isothermal surfaces, and the gradient and heat flow method is identical.

No heat flows within an isothermal surface or curve. If $k \to 0$, $\vec{q} \approx 0$ for a given ∇t; this is a good insulator. As $k \to \infty$, $\nabla T = -\frac{1}{k}\vec{q} \approx 0$; this is a good conductor. In a perfect conductor, there is no variation of temperature. Heat pipes are artificial super conductors. The thermal conductivity can depend on the temperature. For air, it's better conductor when hotter. For iron, it becomes better insulator at hotter temperature. Water, interestingly, has a maximum conductivity at around 400 K, but not at temperatures lower and higher. We typically do not concern with this change since, in most temperature cases, the change in k is not significant.

In a stationary solid, a differential equation for the temperature $T(x, y, z, t)$ can be derived from 2 things: the energy balance from first law of thermodynamics, and the Fourier's law. If we can solve the differential equation, we can predict the temperature distribution; the Fourier's law can also be used to determine the heat flux.

We cut out a subset of a body inside a bigger body, since differential equations are valid inside a body. The first law is used. No net work is done, hence, the energy differential is

$$\frac{\partial \text{energy}}{\partial t} = -\text{net}\frac{\text{energy}}{\text{time}}\text{out} + \frac{\text{energy}}{\text{time}}\text{generated in region} \tag{2}$$

The 'generated heat' could be electricity, chemical reactions, nuclear reactions. Let e be the internal energy per unit volume. Let the net energy out per unit time be $-\int\int \vec{q}\cdot\vec{n}\ \mathrm{d}t$.

$$\frac{\partial}{\partial t}\int\int\int e\,\mathrm{d}V = -\int\int \vec{q}\cdot\vec{n}\ \mathrm{d}t + \int\int\int \dot{q}\,\mathrm{d}v \tag{3}$$

The divergence theorem states that $\int\int \vec{f}\cdot\vec{n}\,\mathrm{d}t = \int\int\int \vec{\nabla}\cdot\vec{f}\,\mathrm{d}V$. This yields the equation

$$\int\int\int \frac{\partial e}{\partial t}\,\mathrm{d}V = -\int\int\int \vec{\nabla}\cdot\vec{q}\,\mathrm{d}V + \int\int\int \dot{q}\,\mathrm{d}V \tag{4}$$

Which yields

$$\int\int\int \left[\frac{\partial e}{\partial t} + \vec{\nabla}\cdot q - \dot{q}\right]\mathrm{d}V = 0 \tag{5}$$

where the bracket term must yield zero:

$$\frac{\partial e}{\partial t} + \vec{\nabla}\cdot\vec{q} - \dot{q} = 0 \tag{6}$$

everywhere in the body. Here, there is no time term. However, notice that e is a function of time. For a solid, the chain rule states that

$$\begin{aligned}\frac{\partial e}{\partial t} &= \frac{\partial e}{\partial T}\frac{\partial T}{\partial t}\\ \frac{\partial e}{\partial t} &= c\frac{\partial T}{\partial t}\end{aligned} \tag{7}$$

where c is the specific heat. Note that for a solid, specific heat of volume and temperature are nearly identical. This now can be introduced to the differential equation to obtain

$$\dot{q} = c\frac{\partial T}{\partial t} + \vec{\nabla}\cdot(-k\vec{\nabla}T) \tag{8}$$

If k is a constant, we can take it out of the gradient, which yields

$$\dot{q} = c\frac{\partial T}{\partial t} - k\vec{\nabla}^2 T \tag{9}$$

where $\vec{\nabla}^2$ is the Laplacian. This equation is often called the diffusion equation, or heat conduction equation; this is a differential equation of the temperature inside the body.

Boundary and initial conditions are required to solve this.

Electromagnetic radiation can come as different wavelengths. Radio waves and microwaves have a certain wavelength. Certain bands of radiation are the ones that cause heating. Table 1.2 has this table, from 0.4 micrometer, up to a 1000 micro meter; these are thermal radiation. Lower pressure could cause sodium to be metal. There could be biological examples of aerogels, or via chemical processes. Diamond isn't considered a ceramic.

This course is *theory* of heat transfer, hence, there will be no experiments, and mostly mathematical modelling.

Simplest case of solving 8 is steady temperature distribution, in a one-dimensional slab, with no sources, a constant k, and simple temperature boundary conditions. A *steady* Distribution indicates non-dependence on time; *transient* Indicates dependence on time. Things could change with space. In a steady condition, time derivatives are out. No sources indicate $\dot{q} = 0$. One dimensional approximations can be taken when spatial variation is primarily on one axis, e.g. a thick wall in the direction of inside to outside. These approximations yield a very simple case:

$$-\cancelto{0}{k}\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} = 0$$

where the solution is

$$T(x) = c_1 x + c_2$$

and the constants can be determined via the boundary condition. If the boundary condition at the origin is T_1, and at the other point is T_2, then the solution is

$$T(x) = \frac{T_2 - T_1}{L}x + T_1$$

Here, the heat flux is obtained by taking the derivative of $T(x)$ w.r.t. x, which yields

$$q_x = -k\frac{\mathrm{d}T}{\mathrm{d}x} = -k\left(\frac{T_2 - T_1}{L}\right) \tag{10}$$

If we wanted the heat per time for our specific area,

$$Q = qA = \frac{kA}{L}(T_1 - T_2) \tag{11}$$

in this simple case. This is analogous to taking temperature as voltage, and Q is the current. Ohm's low over a resistor is $\Delta V = Ri$, which is analogous to

$$(T_1 - T_2) = R_{th}Q = \frac{L}{kA}Q \tag{12}$$

where $T_1 > T_2$. This is what the 'R' value stands for insulation, though they don't include the units.

There will be no class on Monday unless he lets us know. We should try the homework. Office hours are on Tuesdays and Thursdays, 9 AM to 11 AM.

If the conductivity changes by 10%, we'll just have 10% error.

2 Lecture 2

Office hours will now be Tuesday 9 to 11, and Wednesday 1 to 3.

The k is a local conductivity, but we approximate the same everywhere. c is the specific heat and is also local, but we assume it to be the same everywhere. The internal heat generation, $\dot{q}$, can often be ohmic heating.

Assume a slab with constant $\dot{q} > 0$. Let there be a heat flux boundary conduction. Heat transfer is a boundary value problem. Assume a steady $T(x)$ in a 1D slab, with constant k. This means $\frac{\partial T}{\partial t} = 0$, which yields $0 = k\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + \dot{q}$. Once we integrate twice, we get

$$T(x) = -\frac{\dot{q}}{k}\frac{x^2}{2} + c_1 x + c_2$$

where c_1 and c_2 have to be determined by boundary conditions. Suppose $x = 0$ has perfect insulation, so there is no heat transfer: $q(0) = 0$. At $x = L$, we'll have a convection boundary condition $\bar{h}(T(L) - T_\infty) = q(L)$. With the equation and two boundary conditions, we can re-write and get the equations:

$$\begin{aligned} -k\left(-\frac{\dot{q}x}{k} + c_1\right) &= 0 && x = 0 \\ -k\left(-\frac{\dot{q}x}{k} + c_1\right) &= \bar{h}(-\frac{\dot{q}}{k}\frac{x^2}{2} + c_1 x + c_2 - T_\infty) && x = L \end{aligned}$$

The first equation simply yields $c_1 = 0$. In the second equation, we get

$$c_2 = \frac{\dot{q}L}{\bar{h}} + \frac{\dot{q}L^2}{2k} + T_\infty.$$

Plugging this back into the original temperature equation, we obtain

$$T(x) = \dot{q}\left(-\frac{x^2}{2k} + \frac{L}{\bar{h}} + \frac{L^2}{2k}\right) + T_\infty$$

This equation, quite obviously, shows that the temperature is hottest at the insulated wall; the heat transfer (derivative of T) is flat at this region. The derivative is negative at $x = L$, therefore, there is a decreasing temperature at that region.

We can make this plot be dimensionless by making T_∞ be the reference temperature:

$$\frac{T(x) - T_\infty}{\dot{q}L^2/k} = \frac{1}{2} + \frac{k}{\bar{h}L} - \frac{x^2}{2L^2}.$$

Here, $\frac{\bar{h}L}{k} = \mathrm{Bi}$ is the Biot number. The Biot number is useful because, when the Biot number is small < 0.1, the temperature distribution inside the material is fairly constant. When the Biot number is small, the temperature distribution in the body is almost uniform. There will be at most a 5% difference in internal temperature in contrast to the external. This simplifies the problem. A large Biot number creates the opposite problem.

The Biot number can be used for the thermal resistance. $Q = qA$. The conduction resistance between T_L and T_∞ is

$$R_{\text{conv}} = \frac{1}{\bar{h}A} \tag{13}$$

and the resistance between T_O and T_L is

$$R_{\text{cond}} = \frac{L}{kA} \tag{14}$$

and

$$\mathrm{Bi} = \frac{R_{\text{cond}}}{R_{\text{conv}}}. \tag{15}$$

When the Biot number is small, the majority of the temperature drop is on the outside.

If the $\bar{h} \to \infty$ is almost a temperature boundary condition: a larger value indicates that the temperature is closer to the outside ambient temperature. A $\bar{h} = 0$ would indicate insulation on the outer surface as well.

If instead of a slab we had a 1D axis-symmetric cylinder, we would have the same original equations. The Laplacian operator would be in cylindrical coordinates for 1D is:

$$\dot{q} = c\frac{\partial T}{\partial t} - \frac{1}{r}\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}T}{\mathrm{d}r}\right)\right) \tag{16}$$

If there are no sources and time dependence, then

$$T(r) = c_1 \ln r + c_2$$

Suppose we have a problem where $T(r_i) = T_i$ and $T(r_o) = T_o$, then the solution is

$$T(r) = T_i + (T_o - T_i)\frac{\ln r/r_i}{\ln r_o/r_i} \tag{17}$$

with the heat transfer being

$$q(r) = \frac{k(T_i - T_o)}{r\ln(r_o/r_i)}. \tag{18}$$

This may be counter-intuitive since this states that the heat transfer decreases as we go more outside, however, if we consider the net heat transfer Q, we get

$$Q = 2\pi L\frac{k(T_i - T_o)}{\ln(r_o/r_i)}. \tag{19}$$

This is correct: the heat transfer rate, for no heat generation and time dependence, must be the same through all the circles. We can write the thermal resistance as

$$R_{\text{cond}} = \frac{\ln(r_o/r_i)}{2\pi kL}. \tag{20}$$

A thicker layer of insulation increases this resistance.

We can do this technique for a spherical shell. Each spherical layer is isothermal, assuming dependence only on r. Assuming 1D, and taking the Laplacian in spherical coordinates, we have

$$\dot{q} = c\frac{\partial T}{\partial t} - \frac{1}{r^2}\left(\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}T}{\mathrm{d}r}\right)\right). \tag{21}$$

Assuming steady state and no heat generation, the temperature distribution is

$$T(r) = \frac{c_1}{r} + c_2. \tag{22}$$

The temperature boundary conditions are boundary conditions are of the first kind, the heat flux boundary conditions are of the second kind, and having convection involves both and consequently is of the third kind.

Assuming $T(r_i) = T_i$ and $T(r_o) = T_o$, we get

$$T(r) = T_i + \frac{T_i - T_o}{\frac{1}{r_i} - \frac{1}{r_o}}\left(\frac{1}{r} - \frac{1}{r_i}\right) \tag{23}$$

and the heat transfer is

$$q(r) = \frac{k}{r^2} \cdot \frac{T_i - T_o}{\frac{1}{r_i} - \frac{1}{r_o}} \tag{24}$$

and the net heat transfer through each isothermal shell is

$$Q = 4\pi k \cdot \frac{T_i - T_o}{\frac{1}{r_i} - \frac{1}{r_o}}. \tag{25}$$

The thermal resistance is now

$$R_{\text{cond}} = \frac{\frac{1}{r_i} - \frac{1}{r_o}}{4\pi k}. \tag{26}$$

Notice that the above numerator is positive.

3 Lecture 3

For a cooling fin, there is a Biot number in the transitive direction

$$\text{Bi}_{\text{tr}} = \frac{\bar{h}d}{k} \tag{27}$$

and when it is very small, there is hardly any change in temperature about the radius of the fin.

In the cone problem in our homework, we had no heat flux on the sides. Now we can consider it. In steady state, the heat loss differential equation for conduction and convection is

$$0 = \frac{\mathrm{d}}{\mathrm{d}x}\left(-kA(x)\frac{\mathrm{d}T}{\mathrm{d}x}\right) + \bar{h}P(x)\,\mathrm{d}x(T - T_\infty)$$

If k and A are constant, then we have instead

$$kA\frac{\mathrm{d}^2T}{\mathrm{d}x^2} - \bar{h}P(T - T_\infty) \tag{28}$$

which can involve a change of variables. The purpose of the change of variables is to make the math easier for more complex cases. We let T take on $T - T_\infty$, to get a dimensionless

$$\theta = \frac{T - T_\infty}{T_o - T_\infty} \tag{29}$$

and

$$\xi = \frac{x}{L} \tag{30}$$

which now converts the original equation to

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\theta(T_o - T_\infty)\right) - \frac{\bar{h}P}{kA}\theta(T_o - T_\infty) = 0$$

where $T_o - T_\infty$ can be cancelled on both sides, and we can apply the chain rule to convert $\frac{\mathrm{d}\theta}{\mathrm{d}x} \to \frac{\mathrm{d}\theta}{\mathrm{d}\xi}\frac{1}{L}$, and once more, $\frac{\mathrm{d}^2\theta}{\mathrm{d}x^2} \to \frac{1}{L^2}\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2}$. When changing variables in a derivative, we use the chain rule. Note that

$$m^2 \equiv \frac{\bar{h}P}{kA} \tag{31}$$

which now allows the simplification

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}\xi^2} - (m^2L^2)\theta = 0$$

and finally the solution

$$\theta = c_1\exp(mL\xi) + c_2\exp(-mL\xi) \tag{32}$$

where the constants can be found by taking the boundary conditions. Two simple conditions are: temperature at the starting point is equal to the temperature that the fin is touching, and the insulated tip $Q(L) = 0$ condition. Formally, this is: $\theta(\xi = 0) = 1$ and $\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{\xi=1} = 0$. This would yield two equations for the two unknown constants:

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 mLe^{mL} - c_2 mLe^{-mL} &= 0 \end{aligned}$$

which yields the solutions

$$c_1 = \frac{\exp(-mL)}{\exp(mL) + \exp(-mL)} \tag{33}$$

$$c_2 = \frac{\exp(mL)}{\exp(mL) + \exp(-mL)} \tag{34}$$

Note that this is non-dimensional. This can be further simplified by realizing that $\cosh(z) = \frac{e^z + e^{-z}}{2}$:

$$\theta(\xi) = \frac{\cosh(mL(1-\xi))}{\cosh(mL)} \tag{35}$$

The heat transfer at the wall-fin interface in the input energy into the fin, and can be obtained by Fourier's law:

$$\begin{aligned} Q(x=0) &= -KA\frac{\mathrm{d}T}{\mathrm{d}x} \\ &= A\underbrace{km\tanh(mL)}_{\text{fin properties}}(T_o - T_\infty) \\ &= \sqrt{kA\bar{h}P}\tanh(mL)(T_o - T_\infty) \end{aligned}$$

Based on this, the heat transfer with $mL = 3$ and $mL = 5$ is practically same. Heat transfer maxes out at $mL = 3$, and that is the practical length of the useful length.

The heat transfer at the tip of the fin is $Q_L = \bar{h}_L A(T_L - T_\infty)$. Biot number shows up when the equations are in non-dimensional form:

$$-\left.\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right|_{x=L} = \text{Bi}_L = \frac{\bar{h}_L L}{k} \tag{36}$$

The solution for non-dimensional temperature is

$$\theta(\xi) = \frac{\cosh(mL(1-\xi)) + \frac{\text{Bi}_L}{mL}\sinh(mL)(1-\xi)}{\cosh(mL) + \frac{\text{Bi}_L}{mL}\sinh(mL)} \tag{37}$$

where $\frac{\text{Bi}_L}{mL} << 1$ is practically an insulated tip. Heat transfer coefficient at the tip is difficult to calculate. Also, the temperature at the fin start is actually slightly lower than the surrounding body surface. We can compute the real temperature distribution in the fin and in the body near the fin can be done in COMSOL.

If a sphere is in a vacuum, there is no conduction or convection, however radiation still exists.

Unless a body is at 0K, materials will radiate heat: two bodies would radiate to each other. What matters is the difference between the two heat radiation transfers, where the net is the difference of the two:

$$Q_{12,\text{net}} = A_1 F_{1-2} \sigma (T_1^4 - T_2^4) \tag{38}$$

where F is the transfer factor, which depends on the geometry and the surface properties of the two surfaces. The $\sigma = 5.67 \times 10^{-8} \, \frac{\text{W}}{\text{m}^2\text{K}^4}$ is the Stefan-Boltzmann constant. The temperature must be in Kelvin.

If we take a small gray body in an isothermal environment, the transfer factor is equal to the emittance of the small body ϵ_1.

We can simplify the radiation by taking the mean and difference: $T_1 = T_m + \Delta T/2$ and $T_2 = T_m - \Delta T/2$. Plugging these into the radiation equation, we can make it more manageable.

$$\boxed{h_{rad} = 4\sigma T_m^3 \epsilon} \tag{39}$$

where $T_m = (T_1 + T_2)/2$.

Insulated tip may not necessarily have the temperature as the surrounding, especially if $mL \neq 3$. The h locally can be low close to the root.

4 Lecture 4

We can factorize the cubes in radiation to get

$$A_1[\varepsilon_1 \sigma (T_1^2 + T_2^2)(T_1 + T_2)](T_1 - T_2)$$

By introducing mean temperature and temperature difference, we get

$$Q_{12} = A_1 \varepsilon_1 \sigma (T_m^2 + T_m \Delta T + \frac{1}{4}\Delta T^2 + T_m^2 - T_m \Delta T + \frac{1}{4}\Delta T^2) \ldots$$
$$(2T_m)(T_1 - T_2)$$

If we suppose that the temperature difference is small compared to the mean temperature, which is realistic since we use Kelvin, we can approximate to get

$$Q_{12} = A_1 \epsilon_1 \sigma 4 T_m^3 (T_1 - T_2)$$

which finally yields a simple

$$\bar{h}_{\text{rad}} = \varepsilon_1 \sigma \cdot 4 T_m^3 \tag{40}$$

First check Biot numbers to see if we can assume identical temperature. Then, we can do an energy balance. When we are not given the value of mL for a fin, we can realize that heat loss does not get much better after 3, and $\lim_{(mL_w \to \infty) \approx 3} \tanh(mL_w) = 1$. Assuming this, we can also identify the length of the wire for it to be considered a long fin. In the resistor problem for 4.10, we have

$$0.1\,\text{W} = [\bar{h}A + 4\varepsilon\sigma T_m^3 A_2 \sqrt{k_w A_w \bar{h}_w P_w}](T - T_\infty)$$

We can often assume an ε of 0.9, which is close to a blackbody. One of the challenges with this equation is that we don't know what T_m is, since we don't know the body temperature. We are required to *guess* the mean temperature in *Kelvin*.

For transient conduction, the simplest case is a small body with convection boundary condition. Small means the Biot number being small. Doing the energy balance for the body, we can write

$$\frac{\text{d}}{\text{d}t}(e \cdot m) = -\bar{h}A(T - T_\infty) \tag{41}$$

and can also be written as

$$\frac{\text{d}T}{\text{d}t} + \frac{\bar{h}A}{mc}T = \frac{\bar{h}A}{mc}T_\infty(t) \tag{42}$$

where

$$\tau = \frac{mc}{\bar{h}A} \tag{43}$$

is the time constant. We can also assume $T_\infty(t)$ to be a constant. This allows us to have a solution

$$T - T_\infty = C \exp\left(-\frac{t}{\tau}\right) \tag{44}$$

where the constant is found via the initial condition:

$$C = T_i - T_\infty \tag{45}$$

This allows writing as

$$\exp\left(-\frac{t}{\tau}\right) = \frac{T - T_\infty}{T_i - T_\infty}.$$

This can be thought of as a quenching problem since temperature of the body drops until it reaches environmental temperature after a long time. 'Long time' is, for practical purposes, is 5τ.

Now, suppose we have a periodically varying environmental temperature. The simplest way of writing this is

$$T_\infty(t) = T_m + \Delta T \cdot \sin(\omega t). \tag{46}$$

This poses whether the body can follow this temperature. If the time constant is too large, then it will not keep up, and will maintain a rather steady temperature.

We can find the frequency response due to this forced temperature. This involves differential equations:

$$\begin{aligned} \dot{T} + \frac{1}{\tau}T &= \frac{1}{\tau}T_\infty(t) \\ \frac{\text{d}}{\text{d}t}(T - T_m) + \frac{1}{\tau}(T - T_m) &= \frac{\Delta T}{\tau}\sin(\omega t) \\ T - T_m &= (T - T_m)_h + (T - T_m)_p \end{aligned} \tag{47}$$

where $\square_h$ is the general solution, and $\square_p$ is the particular solution. The former, by default, has the transient solution

$$(T - T_m)_h = Ce^{-t/\tau}.$$

This is not too important in the long run, since it decays. The steady state solution is due to the particular solution. This solution is given as

$$(T - T_m)_p = A(\omega)\sin(\omega t - \phi). \tag{48}$$

We can find the missing parameters through complex analysis.

$$\frac{\Delta T}{\tau} = \text{Im}\left[\frac{\Delta T}{\tau}e^{i\omega t}\right]$$

and

$$T - T_m = \text{Im}\left[\mathbb{C}e^{i\omega t}\right]$$

where $\mathbb{C}$ includes the amplitude and phase shift together. Taking these two equations and placing it into the general equation 47:

$$\frac{\Delta T}{\tau} = \text{Im}\left[\mathbb{C}i\omega\cancel{e^{i\omega t}} + \frac{1}{\tau}\mathbb{C}\cancel{e^{i\omega t}}\right]$$

which yields

$$T - T_m = \text{Im}\left[\frac{\Delta T}{1+i\omega\tau}e^{i\omega t}\right]$$

and can be simplified into the more human-readable solution

$$T = T_m + \left(\frac{\Delta T}{\sqrt{1+\omega^2\tau^2}}\right)\sin(\omega t - \phi) \tag{49}$$

where $\phi = \tan^{-1}(\omega\tau)$. When the time constant is much smaller than $1/\omega$, the body will follow the surrounding temperature readily. On the other extreme, as $\omega\tau \to \infty$, then $T(t) \to T_m$.

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c is specific heat. Spherical shell could be a small body with a low Biot number. 'Initial' is arbitrary. τ is time. 5τ is the time after which a body comes to temperature of the surrounding. The fact of $1/e$ coming up in thermo is because of molecular statistical thermodynamics. Some statistical probability distribution approach a continuous Gaussian distribution when the number of particles become large enough. There's some connection between continuum thermo and statistical thermo. The ω is the frequency. The ω in spring mass system is the natural frequency, when it oscillates all by itself. The ω is better thought of as a frequency of an external system. Thermal systems don't oscillate by themselves. Most thermal systems have decay or increase. In mechanics, things oscillate by itself. Same goes for inductors and capacitors.

5 Lecture 5

Chapter 5 involves multidimensional heat transfer. For example, quenching of a semi-infinite slab. Quenching is a sudden change in temperature. The temperature distribution is given by $T(x,t)$, which has two dimensions and hence is multidimensional. In this case, the heat transfer equation is

$$\rho c\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}. \tag{50}$$

Re-arranging this we obtain the thermal diffusivity

$$\alpha = \frac{k}{\rho c} \tag{51}$$

which is a material property and shows up in

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}. \tag{52}$$

The α term can be considered as how fast temperature change can diffuse into a solid.

We can assume an initial condition T_i at $t = 0$ at all $x = 2$. A boundary condition could be, for all $t > 0$, $T(0,t) = T_\infty$. The moment we submerge it, the surface temperature very quickly reaches the temperature of the surrounding. Since this equation is second-order in space, a second boundary condition is needed. This could be that, for $t > 0$, $T(x \to \infty, t) \to T_i$. The solution to this problem will be such that the initial, boundary, and equation, are all solved. We can solve it analytically or computationally: the basic method is identical. First, the problem has to be posed (setting the conditions and parameters).

We will solve this problem using similarity via dimensional analysis. Say that the solution will be of the form

$$T(x,t) = f(x,t,T_\infty,T_i,\alpha).$$

First, we select a reference temperature in this problem, and convert it to dimensionless:

$$\frac{T(x,t)-T_\infty}{T_i - T_\infty} = f(x,t,\alpha)$$

and dimensionless analysis leads to $f(\frac{x}{\sqrt{\alpha t}})$. Let

$$\theta = \frac{T(x,t)-T_\infty}{T_i-T_\infty}$$
$$\zeta = \frac{x}{\sqrt{\alpha t}}$$

and consequently we have a non-dimensionless function $\theta(\zeta)$. The curves at different times in a temperature-space graph yields only one curve if the x axis were to be ζ. We now get the equation for $\frac{\partial T}{\partial t}$:

$$\begin{aligned}\frac{\partial T}{\partial t} &= \frac{\partial}{\partial t}\left[\theta(T_i - T_\infty) + \cancel{T_\infty}\right]\\ &= (T_i - T_\infty)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\frac{\partial \zeta}{\partial t}\\ &= (T_i - T_\infty)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\cancelto{\zeta}{\frac{x}{\sqrt{\alpha t}}}\left(-\frac{1}{2t}\right)\end{aligned} \tag{53}$$

Then, the $\frac{\partial^2 T}{\partial x^2}$ is:

$$\begin{aligned}\frac{\partial^2 T}{\partial x^2} &= \frac{\partial}{\partial x}\left[(T_i - T_\infty)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\frac{\partial\zeta}{\partial x}\right]\\ &= \frac{\partial}{\partial x}\left[(T_i - T_\infty)\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\frac{1}{\sqrt{\alpha t}}\right]\\ &= \frac{1}{\alpha t}(T_i - T_\infty)\frac{\mathrm{d}^2\theta}{\mathrm{d}\zeta^2}\end{aligned} \tag{54}$$

Combining this two into equation 52, we get

$$\begin{aligned}\cancel{\frac{1}{\alpha}(T_i - T_\infty)}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta}\zeta\left(-\frac{1}{2\cancel{t}}\right) &= \cancel{\frac{1}{\alpha t}(T_i - T_\infty)}\frac{\mathrm{d}^2\theta}{\mathrm{d}\zeta^2}\\ \frac{\mathrm{d}^2\theta}{\mathrm{d}\zeta^2} + \frac{\zeta}{2}\cdot\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} &= 0\end{aligned} \tag{55}$$

which is an ordinary differential equation instead of a PDE. Taking the similarity variable can be thought of as converting a PDE to an ODE. We can simplify this using separation of variables:

$$\begin{aligned}
H &\equiv \frac{\mathrm{d}\theta}{\mathrm{d}\zeta} \\
\frac{\mathrm{d}H}{\mathrm{d}\zeta} &= -\frac{\zeta}{2}H \\
\Rightarrow 2\frac{\mathrm{d}H}{H} &= -\zeta\,\mathrm{d}\zeta \\
\Rightarrow H &= C_1 \exp\left(-\frac{\zeta^2}{4}\right)
\end{aligned}$$

and we can plug back in the definition of H and integrate to obtain the solution:

$$\theta = \int_0^\zeta C_1 \exp\left(-\frac{u^2}{4}\right)\, \mathrm{d}u + C_2 \tag{56}$$

The boundary conditions is now $T(x=0,t) = T_\infty \implies \theta(0) = 0$. This means that the C_2 is 0. Then, with the initial condition $T(x,0) = T_i \implies \theta(\infty) = 1$, which then gives us $C_1 = 1/\sqrt{\pi}$. These yield

$$\frac{T(x,t) - T_\infty}{T_i - T_\infty} = \frac{1}{\sqrt{\pi}} \int_0^{\frac{x}{\sqrt{\alpha t}}} e^{-u^2/4}\, \mathrm{d}u \tag{57}$$

This takes the integral under a Gaussian distribution.

We can also take the error function. This is given as

$$\mathrm{erf}(z) \equiv \frac{2}{\sqrt{\pi}} \int_0^\infty e^{-s^2}\, \mathrm{d}s \tag{58}$$

This allows our earlier solution to be

$$\theta = \mathrm{erf}\left(\frac{\zeta}{2}\right). \tag{59}$$

Using this, we can find how far we have to go into the slab before the temperature is unaffected by the quenching. The $\theta = 0.99$ when $\eta/2 > 1.82$. Converting it into dimensional, we have $x = 3.64\sqrt{\alpha t}$, which is a moving boundary or *penetration depth* for the sudden change in surface temperature.

Consequently, for small time, semi-infinite body holds true.

Now, if we had two semi-infinite bodies of different temperatures were brought together, we'd get a temporary infinite slab. We can define the meeting point as the beginning of the coordinate, with one direction being x_1 and other being x_2. For a short enough time, the sudden change will propagate. This creates two equations:

$$\begin{aligned}
\frac{1}{\alpha_1}\frac{\partial T_1}{\partial t} &= \frac{\partial^2 T_1}{\partial x_1^2} \\
\frac{1}{\alpha_2}\frac{\partial T_2}{\partial t} &= \frac{\partial^2 T_2}{\partial x_2^2}
\end{aligned}$$

where the solution will be

$$\begin{aligned}
T_1(x_1,t) &= C_2 + C_1 \mathrm{erf}\left(\frac{x_1}{2\sqrt{\alpha_1 t}}\right) \\
T_2(x_2,t) &= D_2 + D_2 \mathrm{erf}\left(\frac{x_2}{2\sqrt{\alpha_2 t}}\right)
\end{aligned}$$

There is also a thermal interface condition, where $T_1(0,t) = T_2(0,t)$. Each body has their own initial temperatures. The interface condition is given by

$$-k_1\frac{\partial T_1}{\partial x_1} = k_2\frac{\partial T_2}{\partial x_2}. \tag{60}$$

The α is isotropic only if material is isotropic.

6 Lecture 6

For two infinite slabs being brought into contact, we have two coordinates starting from the contact point. Last time, we obtained our solution to be

$$\begin{aligned}
T_1(x_1,t) &= C_2 + C_1 \mathrm{erf}\left(\frac{x_1}{2\sqrt{\alpha_1 t}}\right) \\
T_2(x_2,t) &= D_2 + D_1 \mathrm{erf}\left(\frac{x_2}{2\sqrt{\alpha_2 t}}\right)
\end{aligned}$$

with the initial condition $T_1 = T_{i1}$ at $t = 0$, which forces $C_2 + C_1 = T_{i1}$, and similarly $D_2 + D_1 = T_{i2}$. Additionally, $C_2 = D_2$ since $T_1 = T_2$ at $x_1 = x_2 = 0$. The heat transfer going in either direction must be identical in both bodies. At the interface,

$$-k_2\frac{\partial T_2}{\partial x_2} = -k\frac{\partial T_1}{\partial x_1}$$

at $x_1 = x_2 = 0$. Taking the derivative of T_2 and T_1 for the above equation, we get

$$k_2 D_1 \frac{1}{2\sqrt{\alpha_1 \cancel{t}}} = -k_1 C_1 \frac{1}{2\sqrt{\alpha_1 \cancel{t}}}. \tag{61}$$

Solving for the constants,

$$\begin{aligned}
C_1 &= (T_{i1} - T_{i2})\frac{\frac{k_2}{\sqrt{\alpha_2}}}{\frac{k_1}{\sqrt{\alpha_1}} + \frac{k_2}{\sqrt{\alpha_2}}} \\
D_1 &= (T_{i2} - T_{i1})\frac{\frac{k_1}{\sqrt{\alpha_1}}}{\frac{k_1}{\sqrt{\alpha_1}} + \frac{k_2}{\sqrt{\alpha_2}}} \\
D_2 &= C_2 \\
&= T_{i1}\frac{\frac{k_1}{\sqrt{\alpha_1}}}{\frac{k_1}{\sqrt{\alpha_1}} + \frac{k_2}{\sqrt{\alpha_2}}} + T_{i2}\frac{\frac{k_2}{\sqrt{\alpha_2}}}{\frac{k_1}{\sqrt{\alpha_1}} + \frac{k_2}{\sqrt{\alpha_2}}}
\end{aligned}$$

Knowing these, we are able to find the temperature distribution in any part of the body at any time. The interface takes a constant temperature $T_{\text{interface}} = C_2 = D_2$. This basically states that the interface temperature should be between the initial temperatures of the two bodies. When the interface properties are identical, the interface temperature is the average of the two temperatures.

As an example, beef has $k_1 = 0.48\,\frac{\mathrm{W}}{\mathrm{m\cdot K}}$, $\alpha_1 = 1.35 \times 10^{-7}\,\mathrm{m^2/s}$, with a temperature of $T_{i1} = 20\,°\mathrm{C}$. If the other surface is steel, $k = 60\,\frac{\mathrm{W}}{\mathrm{m\cdot K}}$, $\alpha_2 = 1.88 \times 10^{-5}\,\mathrm{m^2/s}$ at a temperature of $T_{i2} = 80\,°\mathrm{C}$. In this example, the $T_{\text{interface}} = 74.8\,°\mathrm{C}$. When we touch this piece, we'd feel that it is hot. If it was plywood instead with the same temperature, then its interface temperature would be $30.7\,°\mathrm{C}$.

Consequently, thermometers and other sensors try to immediately have the temperature of the item being measured.

For a short time, finite bodies behave as semi-infinite.

Next, we consider a semi-infinite slab with a periodically varying temperature boundary condition:

$$T(0,t) = \bar{T} + \Delta T \cos(\omega t).$$

Note that we can consider this in negative time as well. This can be thought of as daily temperature changes on the surface of the ground. Following the same process of derivation as a regular infinite slab, we have

$$T(x,t) = f(\bar{T}, \Delta T, \omega, \alpha, x, t)$$

where the independent dimensions are the temperature, length, and time, and 6 parameters. Hence, there'll be 3 dimensionless groups since $6-3=3$. There is no unique set of these groups, but there are standard choices. We can set non-dimensional time $\Omega = \omega t$, and non-dimensional temperature

$$\theta = \frac{T(x,t)-\bar{T}}{\Delta T}.$$

Finally, the remaining non-dimensional parameter can be

$$\xi = x\sqrt{\frac{\omega}{2\alpha}}$$

which makes $\theta(\xi,\Omega)$ be the function. Using the same techniques as the semi-finite slab, we get the equation

$$\frac{\partial^2\theta}{\partial\xi^2} = 2\frac{\partial\theta}{\partial\Omega} \tag{62}$$

The boundary condition becomes $\theta = \cos\Omega$ at $\xi = 0$. We can use complex exponent to denote the solution, and just consider the real part to be the solution. Inside the body,

$$\theta = \mathrm{Re}\left\{f(\xi)\exp(i\Omega)\right\} \tag{63}$$

has been assumed, and we also have separated the variables ξ and Ω. Inserting this into our differential equation found earlier, we can solve for $F(\xi)$ assuming that $e^{i\Omega}$ is the known function.

Taking the partial derivative of θ w.r.t. ξ takes the *ordinary* derivative of F since it is a function of only 1 variable:

$$\mathrm{Re}\left\{\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2}\cancel{e^{i\Omega}} = 2Fi\cancel{e^{i\Omega}}\right\}$$

which gets a second-order linear differential equation

$$\frac{\mathrm{d}^2F}{\mathrm{d}\xi^2} - 2iF = 0$$

that yields the common solution

$$F(\xi) = C_1\exp\left(\xi\sqrt{2i}\right) + C_2\exp\left(-\xi\sqrt{2i}\right) \tag{64}$$

Tip! To take the nth root of i, convert it to polar form:

$$\boxed{\sqrt[n]{i} = \exp\left(i\frac{\pi}{2n}\right)} \tag{65}$$

Using this, we can get

$$F(\xi) = C_1\exp\left(\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\xi\right) + C_2\exp\left(-\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)\xi\right)$$

which simplifies to

$$F(\xi) = C_1\exp\left(\xi(1+i)\right) + C_2\exp\left(-\xi(1+i)\right).$$

There's an implicit boundary condition to keep the e^{ξ} from going to infinity as $\xi\to\infty$: $C_1 = 0$. The second term has the decay: the temperature far from the surface should stay small.

When $\xi = 0$, $F(\xi) = C_2$, and since $\theta(\xi = 0,\Omega) = \mathrm{Re}\left\{C_2\exp(i\Omega)\right\}$, but one of our boundary condition is $\theta(\xi = 0,\Omega) = \mathrm{Re}\left\{e^{i\Omega}\right\}$, then $C_2 = 1$. Converting

$$\theta = \mathrm{Re}\left\{e^{-\xi}e^{-i\xi}e^{i\Omega}\right\}$$

into the real form, we have the equation **5.62** from the book:

$$\theta = e^{-\xi}\cos(\Omega-\xi). \tag{66}$$

Converting this into dimensional form, we have

$$\frac{T(x,t)-\bar{T}}{\Delta T} = \exp\left(-x\sqrt{\frac{\omega}{2\alpha}}\right)\cos\left(\omega t - x\sqrt{\frac{\omega}{2\alpha}}\right). \tag{67}$$

Notice that this illustrates a propagating wave in a material. The phase is different for different points. Deeper penetration obtained by larger α and smaller ω.

The complete phase shift point is when

$$x\sqrt{\frac{\omega}{2\alpha}} = \pi \tag{68}$$

and should be when $\xi = \pi$.

There may not be a way to make this critically damped.

7 Lecture 7

Semi-infinite solid with heat flux boundary conditions have the same differential condition. We'd need initial condition (uniform temperature) and boundary conditions:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$$

with T_i for $t<0$, and constant flux q_w at $x=0$. The procedure is to first derive a solution with the heat flux. We formulate a problem for $q(x,t)$. This is found by taking the differential equation, and converting it to an equation for q:

$$\begin{aligned}
-k\frac{\partial}{\partial x}\left(\frac{1}{\alpha}\frac{\partial T}{\partial t}\right) &= -k\frac{\partial}{\partial x}\left(\frac{\partial^2 T}{\partial x^2}\right)\\
\frac{1}{\alpha}\frac{\partial}{\partial t}\left(-k\frac{\partial T}{\partial x}\right) &= \frac{\partial^2}{\partial x^2}\left(-k\frac{\partial T}{\partial x}\right)\\
\frac{1}{\alpha}\frac{\partial q}{\partial t} &= \frac{\partial^2 q}{\partial x^2}
\end{aligned}$$

where we can now non-dimensionalize as

$$\hat{\Theta} \equiv \frac{q - q_w}{-q_w}$$

with the input being $\xi \equiv \frac{x}{\sqrt{\alpha t}}$ which forces the differential equation, in terms of q to become:

$$-\frac{\xi}{2}\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = \frac{\mathrm{d}^2\Theta}{\mathrm{d}\xi^2}$$

and the boundary conditions are $q\Big|_{t=0} = 0 \implies \hat{\Theta}(\xi = \infty)1$ and $q\Big|_{x=0} = q_w \implies \hat{\Theta}(\xi = 0) = 0$. This yields

$$\hat{\Theta} = \mathrm{erf}\left(\frac{\xi}{2}\right) \tag{69}$$

which now can be plugged in to solve for q:

$$\begin{aligned}\frac{q - q_w}{-q_w} &= \mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)\\ q(x,t) &= q_w - q_w\mathrm{erf}\left(\frac{x}{2\sqrt{\alpha t}}\right)\\ &= q_w\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)\end{aligned}$$

Since $q = -k\frac{\partial T}{\partial x}$, we can integrate

$$\frac{\partial T}{\partial x} = -\frac{1}{k}q_w\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right)$$

to get

$$T(x,t) = -\frac{q_w}{k}x\mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha t}}\right) + \frac{2q_w}{k\sqrt{\pi}}\sqrt{\alpha t}\exp\left(-\frac{x^2}{4\alpha t}\right) + T_i \tag{70}$$

for the boundary condition

$$T(x=0) = T_i + \frac{2q_w\sqrt{\alpha t}}{k\sqrt{\pi}}.$$

In the case of a semi-infinite solid with convection boundary condition, we have a more realistic problem. The differential equation is identical, the initial condition is $T(x,0) = T_i$, and the environmental temperature is gradually changed to T_∞. This is more realistic because the earlier problem assumes sudden change in body temperature, which does not happen. We get the boundary condition

$$\bar{h}\left(T_\infty - T(x,t)\right) = -k\frac{\partial T}{\partial x} \,@\, x = 0 \tag{71}$$

The solution can be found in 5.6 of the textbook *Conduction of Heat in Solids*, 2nd edition, 1959. The non-dimensional Θ for the solution is

$$\boxed{\Theta = \frac{T - T_\infty}{T_i - T_\infty} = \mathrm{erf}\left(\frac{\xi}{2}\right) + \exp\left(\beta\xi + \beta^2\right)\mathrm{erfc}\left(\frac{\xi}{2} + \beta\right)} \tag{72}$$

where $\beta = \frac{\bar{h}\sqrt{\alpha t}}{k}$. Notice that, if we set $h \to \infty$, then this becomes a constant conduction problem, since surface temperature is forced to become T_∞ instantly. The temperature at the surface, as a function of time, is

$$\frac{T(0,t) - T_\infty}{T_i - T_\infty} = \exp\left(\frac{\bar{h}^2\alpha t}{k^2}\right)\mathrm{erfc}\left(\frac{\bar{h}\sqrt{\alpha t}}{k}\right) \tag{73}$$

In a finite slab with temperature initial condition of $T(x,0) = T_i$ and boundary condition of $T(\pm L, t) = T_\infty$. In semi-infinite body, there is no length scale, and consequently the x and t can be combined into ξ, but now we have a length limitation in a finite body. We have to obtain a new non-dimensional value:

$$\begin{aligned}T - T_\infty &= f(T_i - T_\infty, x, t, \alpha, L)\\ \frac{T - T_\infty}{T_i - T_\infty} &= f\left(\frac{x}{L}, \frac{\alpha t}{L^2}\right)\end{aligned}$$

notice that the Fourier number is given by the dimensionless time

$$\boxed{\mathrm{Fo} \equiv \frac{\alpha t}{L^2}} \tag{74}$$

and, interestingly, the dimensionless space is $\xi = \frac{x}{L} + 1$. We now have the non-dimensional PDE

$$\frac{\partial\Theta}{\partial\mathrm{Fo}} = \frac{\partial^2\Theta}{\partial\xi^2} \tag{75}$$

with initial condition of $\Theta(\xi, \mathrm{Fo} = 0) = 1$ and the boundary condition is $\Theta(\xi = 1 \pm 1, \mathrm{Fo}) = 0$.

We solve this via separation of variables. We assume that the solution is in the form $\Theta(\xi, \mathrm{Fo}) = f(\xi)g(\mathrm{Fo})$. The differential equation can be re-written in this form and solved:

$$\begin{aligned}f(\xi)\frac{\mathrm{d}g}{\mathrm{dFo}} &= \frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2}g(\mathrm{Fo})\\ \Rightarrow \frac{1}{g(\mathrm{Fo})}\frac{\mathrm{d}g}{\mathrm{dFo}} &= \frac{1}{f(\xi)}\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} = -\lambda^2\end{aligned}$$

which yields two equations

$$\begin{aligned}&\frac{\mathrm{d}g}{\mathrm{dFo}} + \lambda^2 g = 0\\ &\Rightarrow g(\mathrm{Fo}) = C\exp\left(-\lambda^2\frac{\alpha t}{L^2}\right)\\ &\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \lambda^2 f(\xi) = 0\\ &\Rightarrow f(\xi) = A\cos\lambda\xi + B\sin\xi\end{aligned}$$

Here, we notice that $\lambda < 0$ since, otherwise, $g(\mathrm{Fo}) \to \infty$ as $t \to \infty$. We can now impose the boundary condition to get $f(0) = 0 \Rightarrow A = 0$, and $f(2) = 0 \Rightarrow B\sin(\lambda \cdot 2)$. In the latter, either $B = 0$ which yields a trivial solution, or $\lambda_n = \frac{n\pi}{2}\,\forall n \in \mathbb{N}$. Note that we choose positive integers because earlier we imposed that ?. We have to sum all the n solutions to obtain the true solution, since not a single solution will give us the correct result. We have to use a Fourier series to obtain our initial condition. First, we combine all the equations:

$$\Theta = \sum_{n=1}^{\infty}\underbrace{BC}_{a_n}\sin(\lambda_n\xi)\exp\left(-\lambda_n^2\mathrm{Fo}\right)$$

and impose the initial condition of $\Theta(\xi, 0) = 1$:

$$1 = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{2}\xi\right)$$
$$\sin\frac{m\pi}{2} = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi\xi}{2}\right)\sin\frac{m\pi\xi}{2}$$
$$\int_0^2 \sin\frac{m\pi\xi}{2}\,d\xi = \int_0^2 \sum_{n=1}^{\infty} a_n \sin\frac{n\pi\xi}{2}\sin\frac{m\pi\xi}{2}\,d\xi$$

and now we can use the orthogonality relation that the integral of $\sin n\xi$ multiplied with $\sin m\xi$ yield 0 if $m \neq n$ and 1 if $m = n$.

$$\int_0^2 \sin\frac{mn\xi}{2}\,d\xi = G_m$$
$$-\left[\frac{2}{m\pi}\cos\frac{m\pi\xi}{2}\right]_{\xi=0}^{\xi=2} = G_m$$
$$-\frac{2}{m\pi}\left[\cos m\pi - 1\right] = G_m$$

where we now notice that for m even numbers, $G_m = 0$, and for m odd numbers, we have $G_m = \frac{4}{m\pi}$. The solution is now

$$\boxed{\Theta(\xi, \mathrm{Fo}) = \sum_{n=1,3,5,\dots}^{\infty} \frac{4}{n\pi}\exp\left(-\left(\frac{n\pi}{2}\right)^2\mathrm{Fo}\right)\sin\frac{n\pi\xi}{2}} \tag{76}$$

which is a useful solution if the terms in the series get small quickly, because they are fastly converging and we can take a couple of terms to get an approximate solution.

8 Lecture 8

For infinite fins, we have to throw out one of the constants. However, for finite problems, keep it as is. The exam will be on March 4th. For a finite slab with length L on *both* sides, with a convective boundary condition with $\bar{h}$ and T_∞ defined. The temperature, initially everywhere, is T_i. The governing equation is $\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2}$. Here, we had the following non-dimensional entities

$$\xi = \frac{x}{L}$$
$$\mathrm{Fo} = \frac{\alpha t}{L^2}$$
$$\Theta = \frac{T - T_\infty}{T_i - T_\infty}$$

which now yields the conditions

$$\frac{\partial \Theta}{\partial \mathrm{Fo}} = \frac{\partial^2\Theta}{\partial\xi^2}$$
$$\Theta(\xi, \mathrm{Fo} = 0) = 1$$
$$\Theta(\xi = 0, 2, \mathrm{Fo}) = 0$$

where we now, by assuming $\Theta = f(\mathrm{Fo})g(\xi)$, we get the constant

$$\frac{\frac{\mathrm{d}f}{\mathrm{dFo}}}{f(\mathrm{Fo})} = \frac{\frac{\mathrm{d}^2 g}{\mathrm{d}\xi^2}}{g(\xi)} = -\lambda^2$$

yields the two equations

$$\frac{\mathrm{d}g}{\mathrm{dFo}} + \lambda^2 g = 0$$
$$\Rightarrow g(\mathrm{Fo}) = C\exp\left(-\lambda^2\mathrm{Fo}\right)$$
$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \lambda^2 f(\xi) = 0$$
$$\Rightarrow f(\xi) = A\cos\lambda\xi + B\sin\xi.$$

With the boundary condition $g(\xi = 0, 2) = 0$ yields $B = 0$ and

$$A\sin 2\lambda = 0 \Rightarrow \lambda_n = \frac{n\pi}{2}\ \forall n \in \mathbb{N}.$$

The coefficients where n is even produces anti-symmetric functions about the center-line, which cancels itself out.

We then have to solve for

$$1 = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi\xi}{2}$$

by multiplying both sides with another sine and then taking the integral:

$$\int_0^2 \sin\frac{m\pi\xi}{2}\,d\xi = \sum_{n=1}^{\infty} a_n \underbrace{\int_0^2 \sin\frac{n\pi\xi}{2}\sin\frac{m\pi\xi}{2}\,d\xi}_{\delta_{mn}}$$
$$\text{where } \delta_{mn} \equiv \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$
$$\int_0^2 \sin\frac{m\pi\xi}{2}\,d\xi = \sum_{n=1}^{\infty} a_n\delta_{mn}$$
$$= a_1\delta_{m1} + a_2\delta_{m2} + \dots$$
$$\Rightarrow -\frac{2}{m\pi}\left[\cos m\pi - 1\right] = a_m$$
$$a_m = \begin{cases} 0 & m = 2i\,\forall i \in \{\mathbb{N}, 0\} \\ \frac{4}{m\pi} & m = 2i + 1\,\forall i \in \mathbb{N} \end{cases}$$

and finally, we get

$$\Theta(\xi, \mathrm{Fo}) = \frac{T - T_\infty}{T_i - T_\infty} \tag{77}$$
$$= \boxed{\sum_{n=1,3,5,\dots}^{\infty} \frac{4}{n\pi}\exp\left(-\left(\frac{n\pi}{2}\right)^2\left(\frac{\alpha t}{L^2}\right)\right)\sin\frac{n\pi}{2}\left(\frac{x}{L} + 1\right)}. \tag{78}$$

This series converges well due to the exponent term.

Now, we'll look at a sphere instead, where our conditions are $T(r, 0) = T_i$ and $T(r_o, t) = T_\infty$ with the governing PDE

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) \tag{79}$$

and we have slightly different non-dimensional terms $\rho = \frac{r}{r_o}$ and $\mathrm{Fo} = \frac{\alpha t}{r_o^2}$. The differential equation in dimensionless form is

$$\frac{\partial\Theta}{\partial\mathrm{Fo}} = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial\Theta}{\partial\rho}\right) \tag{80}$$

where we can again obtain a constant from the separation of variables of $\Theta = f(\text{Fo})g(\rho)$ yielding

$$\frac{\frac{f}{\text{Fo}}}{f(\text{Fo})} = \frac{\frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}g}{\mathrm{d}\rho}\right)}{g(\rho)} = -\lambda^2$$

yielding the two equations and their solutions:

$$\begin{aligned}
\frac{df}{d\text{Fo}} + \lambda^2 f &= 0\\
f(\text{Fo}) &= c_3\text{exp}\left(-\lambda^2\text{Fo}\right)\\
\Rightarrow \frac{1}{\rho^2}\frac{\mathrm{d}}{\mathrm{d}\rho}\left(\rho^2\frac{\mathrm{d}g}{\mathrm{d}\rho}\right) + \lambda^2 g &= 0\\
\Rightarrow g(\rho) &= c_1\frac{\sin\lambda\rho}{\rho} + c_2\frac{\cos\lambda\rho}{\rho}
\end{aligned}$$

We can immediately notice that c_2 must be 0 because $\cos(\lambda\rho)/\rho$ goes to infinity for $\rho = 0$, which should not be possible. Then, we can implement the other conditions of $\Theta(1, \text{Fo}) = 0$ and $\Theta(0, \text{Fo}) > 0$. These yields

$$\Theta = c_3\text{exp}\left(-(n\pi)^2\text{Fo}\right)c_1\frac{\sin n\pi\rho}{\rho}$$

is a solution that satisfies the boundary condition, and we also have

$$\Theta = \sum_{n=1}^{\infty} a_n\text{exp}\left(-(n\pi)^2\text{Fo}\right)\frac{\sin(n\pi\rho)}{\rho}$$

that is a solution. If we attempted to perform

$$1 = \sum_{n=1}^{\infty} a_n\frac{\sin(n\pi\rho)}{\rho}$$

we'd have a slight difficulty with the $1/\rho$ part. We now have

$$\begin{aligned}
\frac{\sin m\pi\rho}{\rho} &= \sum_{n=1}^{\infty} a_n\frac{\sin m\pi\rho}{\rho}\frac{\sin n\pi\rho}{\rho}\\
\int_0^1 \frac{\sin m\pi\rho}{\rho}\rho^2\,d\rho &= \sum_{n=1}^{\infty} a_n\int_0^1 \frac{\sin m\pi\rho}{\rho}\frac{\sin n\pi\rho}{\rho}\,d\rho\\
&= \sum_{n=1}^{\infty} a_n\begin{cases}0 & m \neq n\\ \frac{1}{2} & m = n\end{cases}\\
-\frac{(-1)^m}{m\pi} &= \frac{1}{2}a_n
\end{aligned}$$

which yields us the solution

$$\boxed{\Theta(\rho, \text{Fo}) = \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\text{exp}\left(-(n\pi)^2\text{Fo}\right)\frac{\sin n\pi\rho}{\rho}} \tag{81}$$

Now, we'll do convective boundary condition for a finite slab. We have almost identical of everything, except that now we have the boundary conditions:

$$\bar{h}(T_\infty - T) = -k\frac{\partial T}{\partial x}\bigg|_{x=\pm L}$$

and this enables us to use symmetry, since the heat loss is same on both sides:

$$-k\frac{\partial T}{\partial x}\bigg|_{x=0} = 0$$

and this *voids* one of the sides of the convective boundary condition!

Applying the boundary conditions, we get

$$\bar{h}\left(\cancel{T_\infty} - \Theta(T_i - T_\infty) - \cancel{T_\infty}\right) = -k\frac{\partial(\Theta(T_i - T_\infty) + \cancel{T_\infty}}{\partial\xi}\frac{1}{L}\bigg|_{\xi=0}$$

which simplifies to a typical non-dimensional boundary condition

$$\frac{\partial\Theta}{\partial\xi} = \text{Bi}\Theta\bigg|_{\xi=0} \tag{82}$$

and we also have the symmetry condition which is simply

$$\frac{\partial\Theta}{\partial\xi}\bigg|_{\xi=1} = 0$$

In the exam, we can have hand written notes and book.

9 Lecture 9

The homework is due on March 4th, which is the date of the exam. The homework will have computations. It is recommended that we do the homework by Monday to bring questions. The exam involves HW 1–5.

Last class we did a convection boundary problem on both sides of a semi-infinite lab. There, the boundary condition only applied to $f(\xi)$ which is what turned one of the trigonometric functions to be 0, for $\xi = 0$. For the condition at $\xi = 1$, we have

$$A\lambda\sin(\lambda) = \text{Bi}A\cos(\lambda)$$

which simplifies to

$$\cot\lambda = \frac{\lambda}{\text{Bi}}$$

which is a transcendental function. Notice that this equation describes the intersects between the straight line λ/Bi and the repeating function $\cot\lambda$. As $\lambda \to \infty$, $\lambda = n\pi$.

Applying the initial condition provided

$$1 = \sum_{n=1}^{\infty} A_n\cos\lambda_n\xi$$

which is a generalized Fourier equation. Using the orthogonality equation,

$$\int_0^1 \cos\lambda_m\xi\cos\lambda_n\xi\,d\xi = \begin{cases}0 & m \neq n\\ \frac{\lambda_m + \sin\lambda_m\cos\lambda_m}{2\lambda_m} & m = n\end{cases}$$

which plugged in yields

$$\frac{\sin\lambda_m}{\lambda_m} = A_m\frac{\lambda_m + \sin\lambda_m\cos\lambda_m}{2\lambda_m}$$

The book tables are not recommended; it is recommended that we find the λ_m ourselves. Table 5.1 gives series solutions, for convection boundary conditions.

Now we'll consider finite rectangular and finite-length cylinders.

First, finite rectangle. There is an initial condition of $T(x, y, t = 0) = T_i$ and boundary conditions of

$$-k\frac{\partial T}{\partial x} = \bar{h}(T - T_\infty) \text{ @ } x = \pm L_x$$
$$-k\frac{\partial T}{\partial y} = \bar{h}(T - T_\infty) \text{ @ } y = \pm L_y$$

and the differential equation is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}.$$

With the dimensionless temperature being

$$\Theta = \frac{T(x, y, t) - T_\infty}{T_i - T_\infty}$$

and substituting this into the differential equation yields

$$\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \Theta}{\partial x^2} + \frac{\partial^2 \Theta}{\partial y^2}.$$

We will use the ansantz that $\Theta(x, y, t) = \Theta_x(\xi_x, \mathrm{Fo}_x)\Theta_y(\xi_y, \mathrm{Fo}_y)$ with $\xi_i = i/L_i$, $\mathrm{Bi}_i = \bar{h}L_i/k$ and $\mathrm{Fo}_i = \alpha t/L_i^2$ for $i = x, y$. By doing this, we realize that each individual Θ_i is a solution of a semi-finite slab.

We can validate the differetial equation by

$$\frac{1}{\alpha}\left[\frac{\partial \Theta_x}{\partial t}\Theta_y + \Theta_x\frac{\partial \Theta_y}{\partial t}\right] \overset{?}{=} \frac{\partial^2 \Theta_x}{\partial x^2}\Theta_y + \Theta_x\frac{\partial^2 \Theta_y}{\partial y^2}$$

We can extend this to a finite rectangular prism.

Next, for a finite-length cylinder, the differential equation is a mix of cylindrical and cartesian:

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)$$

and the non-dimensional temperature is the product of the two solutions

$$\Theta(x, r, t) = \Theta_x(\xi_x, \mathrm{Fo}_x)\,\Theta_r\left(\frac{r}{r_o}, \frac{\alpha t}{r_o^2}\right).$$

This requires Bessel functions for the cylindrical component. Note that we'll consider $\bar{h}$ on *all* surfaces, including the flat and curved surfaces.

Chapter 6 is about fluids and convection. It'll involve continuity and linear momentum equation for fluids, and we should review it.

As an example, suppose we move a fin in space through the air at a constant velocity. An example of this is extruding molten metal. We have to do an energy balance at a differential control volume. The coordinates will become spatial: it's no longer attached to a material. This is also called an Eulerian coordinate.

For a control volume,

$$\frac{\partial e}{\partial t} = \frac{e_{in}}{\delta t} - \frac{e_{out}}{\delta t}$$

where e is energy density. We now write this as

$$\begin{aligned}\frac{\partial}{\partial t}&(\rho A\,\mathrm{d}x e)\\ &= Q(x) - Q(x + \mathrm{d}x) - \bar{h}P\,\mathrm{d}x(T - T_\infty)\\ &\quad + \frac{\rho AU\,\mathrm{d}te(x)}{\mathrm{d}t} - \frac{\rho AU\,\mathrm{d}te(x + \mathrm{d}x)}{\mathrm{d}t}.\end{aligned} \tag{83}$$

Here, $Q(x) = -kA\frac{\partial T}{\partial x}$ which makes

$$Q(x + \mathrm{d}x) = Q(x) + \frac{\partial Q}{\partial x}\,\mathrm{d}x = -kA\frac{\partial T}{\partial x} - kA\frac{\partial^2 T}{\partial x^2}\,\mathrm{d}x.$$

The only surviving term between the two Q is the spatial derivative, which now plugged into the energy balance differential equation:

$$\rho A\frac{\partial e}{\partial t}\cancel{\mathrm{d}x} = kA\frac{\partial^2 T}{\partial x^2}\cancel{\mathrm{d}x} - \bar{h}P\cancel{\mathrm{d}x}(T - T_\infty) - \rho AU\frac{\partial e}{\partial x}\cancel{\mathrm{d}x}$$

which yields the familar fluid mechanics equation

$$\rho A\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)e = kA\frac{\partial^2 T}{\partial x^2} - \bar{h}P(T - T_\infty) \tag{84}$$

which involves the material derivative. Notice that, without convection, the LHS is 0, and the equation is a static fin equation. Also, notice that

$$\frac{\partial e}{\partial t} = \frac{\partial e}{\partial T}\frac{\partial T}{\partial t} = c\frac{\partial T}{\partial t}$$

and the same is true for in terms of x.

Spherical egg.

T_{sfc}?

For 5.35 use 3D.

10 Lecture 10

Exam on Wednesday, we can bring notes, and no calculator needed. Test will cover 1–5 (not 3, and 5.7). We should use non-dimensional methods when there's a lot of variables. Most problems will be dimensional. We should compute with the infinite series on the homework.

We can come to his office tomorrow before 1530.

In a fluid flow, there's a velocity vector at any point in space $V(x, y, t)$. We can take the projection on the x axis to get u and on the y axis to get v, which enables us to write the velocity w.r.t. basis vector $V = u\hat{i} + v\hat{j}$. Based on the continuity equation:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla \cdot V) = 0 \tag{85}$$

which holds true in many cases, because this is the fundamental conservation of mass. Separating the material derivative and the divergence in 2D, the above equation now becomes

$$\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right).$$

Note that, if density is constant, we get the simplified equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}.$$

Note that $\nabla \cdot V$ is the volumetric strain. Next, the linear momentum equation, here, has 2 components. The stresses and body forces are equal to the linear momentum in any direction:

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \rho g_x \tag{86}$$

$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \rho g_y. \tag{87}$$

Then, we have the angular momentum principle, which simply states that

$$\tau_{xy} = \tau_{yx}. \tag{88}$$

The classical ideal fluid is a Newtonian fluid where the relations are

$$\sigma_{xx} = -p + 2\mu \frac{\partial u}{\partial x} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{89}$$

$$\sigma_{yy} = -p + 2\mu \frac{\partial v}{\partial y} + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{90}$$

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \tag{91}$$

Then, the continuity equation yields

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = -\frac{\partial \rho}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \dots + (\mu + \lambda) \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{92}$$

$$\rho \frac{\mathrm{D}v}{\mathrm{D}t} = -\frac{\partial \rho}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \dots + (\mu + \lambda) \frac{\partial}{\partial y} \left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} \right) \tag{93}$$

$$\Rightarrow \rho \frac{\mathrm{D}V}{\mathrm{t}} = -\nabla p + \mu \nabla^2 V + (\mu + \lambda) \nabla (\nabla \cdot V) \tag{94}$$

which is the Navier-Stokes equation. Note that the last term is zero for an incompressible fluid.

Next, we need the energy equation for a fluid. The net energy change in time is the net change in heat, work, and flow, per time. The energy density is

$$e = \frac{\text{energy}}{\text{mass}} = \hat{u} + \frac{u^2 + v^2}{2}$$

where $\hat{u}$ is internal energy per unit mass. Note that we'll not consider chemical or nuclear energy. The net heat in any given control volume is

$$\boxed{2} - \frac{\partial Q_x}{\partial x} \,\mathrm{d}x - \frac{\partial Q_y}{\partial y} \,\mathrm{d}y - \frac{\partial}{\partial x} \left(-k \,\mathrm{d}y \,\mathrm{d}z \frac{\partial T}{\partial x} \right) \mathrm{d}x - \frac{\partial}{\partial y} \left(-k \,\mathrm{d}x \,\mathrm{d}z \frac{\partial T}{\partial y} \right) \mathrm{d}y \tag{95}$$

where k could be a function of temperature. Re-arranging this and assuming constant k, we can bring it out. Next, the rate of work is given as a combination of work done by the stresses, shear stresses, and body force. The work done by stress, say in the x direction, would be $(\sigma_{xx} \,\mathrm{d}y \,\mathrm{d}z) \times u$. The work done by shear stress, say in the vertical direction, would be given as $(\tau_{xy} \,\mathrm{d}y \,\mathrm{d}z) v$. Then, the body force work, say in the x direction, would be given as $g_x \rho \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \times u$. Finally, combining all these, we get

$$\boxed{3} = \left[\frac{\partial}{\partial x}(\sigma_{xx} u) + \frac{\partial}{\partial y}(\sigma_{yy} v) \dots + \frac{\partial}{\partial x}(\tau_{xy} v) + \frac{\partial}{\partial y}(\tau_{xy} u) + \rho g_x u + \rho g_y u \right] \mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z \tag{96}$$

which can be simplified as

$$\boxed{3} = (V \nabla \sigma + V^\top \vec{g}) \,\mathrm{d}x \,\mathrm{d}y \,\mathrm{d}z.$$

We can also modify this to write it in terms of viscosity, since stretching and pulling against friction generates heat:

$$\boxed{3} = u\rho \frac{\mathrm{D}u}{\mathrm{D}t} + v\rho \frac{\mathrm{D}v}{\mathrm{D}t} + 2\mu \left[\left(\frac{\partial u}{\partial x} \right)^2 + \left(\frac{\partial v}{\partial y} \right)^2 \right] \dots + \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)^2 + \lambda \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)^2 \tag{97}$$

where the last term may be zero.

No internet access.

11 Lecture 11

With a fluid flow, there is a velocity vector V at every point. The net change in energy per time for a control volume is the sum of three terms:

1. Net heat in
2. Net work on control volume
3. Net energy flow in

e denotes specific internal energy, which is equal to $e = \hat{u} + \frac{1}{2}(u^2 + v^2)$. The equation we derived last time was

$$\underbrace{\frac{\partial}{\partial t}(\rho e)}_{\text{description 0}} = \underbrace{k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right)}_{\text{description 1}} \underbrace{+ \rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\frac{u^2 + v^2}{2} \right) - p \cancel{\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)} + \Phi}_{\text{description 2}} \underbrace{- \frac{\partial}{\partial x}(\rho u e) - \frac{\partial}{\partial y}(\rho v e)}_{\text{description 3}} \tag{98}$$

which is the energy equation. By taking continuity, assuming incompressibility, and relating specific internal energy to the temperature as specific enthalpy $\hat{h} = \hat{u} + \frac{p}{\rho}$, we get

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(\hat{h} - \frac{p}{\rho} \right) = k \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} \right) + \Phi.$$

Note that Φ is the viscous dissipation. For an ideal gas, $\hat{h}$ is a function of temperature only. For other gases at subsonic speeds and many liquids, $\hat{h} \approx \hat{h}(T)$. Material derivative always means the time derivative added with the convective terms. In this case,

$$\frac{\mathrm{D}\hat{h}}{\mathrm{D}t} = \frac{\partial \hat{h}}{\partial t} + u\frac{\partial \hat{h}}{\partial x} + v\frac{\partial \hat{h}}{\partial y}$$

and applying the chain rule, we get

$$\begin{aligned}\frac{\mathrm{D}\hat{h}}{\mathrm{D}t} &= \frac{\mathrm{d}\hat{h}}{\mathrm{d}T}\frac{\partial T}{\partial t} + u\frac{\mathrm{d}\hat{h}}{\mathrm{d}T}\frac{\partial T}{\partial x} + v\frac{\mathrm{d}\hat{h}}{\mathrm{d}T}\frac{\partial T}{\partial y} \\ &= c_p\frac{\mathrm{D}T}{\mathrm{D}t}\end{aligned}$$

Plugging this back in to our earlier equation, we get

$$\rho C_p \frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi \tag{99}$$

which is slightly different from the conduction equation.

Suppose we have a fluid flow over a thin flat plate with constant temperature T_w that is in the same orientation as the fluid flow. Heat will transfer q_w from the plate to the fluid. This can be written as

$$q_w = -k\frac{\partial T}{\partial y}$$

which requires us knowing the temperature. We'll assume steady flow with a flow rate and temperature of u_∞ and T_∞ respectively. A steady flow means that change w.r.t. time is zero. We'll also assume incompressibility.

The continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

and the x-momentum equation is

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

with the corresponding y-momentum being

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right).$$

The energy term is

$$\rho c_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}\right) + \Phi.$$

There are 4 unknowns here, and 4 equations. The boundary conditions are:

- No slip boundary condition at the surface, so $u = v = 0$ on the surface.
- Plate temperature boundary condition of $T = T_w$
- Not used, but far away, $T = T_\infty$

This is a mathematically unsolved problem. Experimentally, there's a viscous boundary layer that forms on the surface of the plate. The length of this is given by δ. When the Reynolds number,

$$\mathrm{Re}_L = \frac{\rho u_\infty L}{\mu} = \frac{u_\infty L}{\nu} \tag{100}$$

is large, the flow outside the boundary layer is inviscid. That is, the viscosity outside the boundary layer is not important. The $\delta / L << 1$. Outsidet his boundary layer, flow is undisturbed. Inside the boundary layer, we can simplify the fluid equations. This was introduced by Prandtl, and he brought about boundary layer theory.

The fluid velocity will range from $u \in [0, u_\infty]$, and this changes as we progress from leading edge to end point L. Additionally, the vertical velocity changes over the boundary layer thickness. As a result, the continuity equation is

$$\frac{u_\infty}{L} + \frac{v}{\delta} \approxeq 0$$

which allows us to re-write the vertical component of velocity to be

$$v \approxeq u_\infty \frac{\delta}{L}.$$

Notice that, here, $\delta << L$, so v will be pretty close to zero: $v \approxeq 0$.

Plugging this simplification to the momentum equations, we get the simplification

$$\begin{aligned}\rho u_\infty \frac{u_\infty}{L} + \rho u_\infty \frac{\delta}{L}\frac{u_\infty}{\delta} &= -\frac{\partial p}{\partial x} + \mu\left(\cancelto{\approx 0}{\frac{u_\infty}{L^2}} + \frac{u_\infty}{\delta^2}\right) \\ \rho\left(u_\infty\frac{u_\infty}{L}\frac{\delta}{L} + u_\infty\frac{\delta}{L}\frac{u_\infty}{\delta}\frac{\delta}{L}\right) &= -\frac{\partial p}{\partial y} + \mu\left(u_\infty\frac{\delta}{LL^2} + u_\infty\frac{\delta}{L\delta^2}\right)\cdot u_\infty\frac{\delta}{L}\end{aligned}$$

Notice that every term in the y-momentum equation is very close to zero. Consequently, we only need to focus on the x-momentum equation. To remove the y-momentum equation, we must assume that, inside the boundary layer, $\frac{\partial p}{\partial y}$ is small, especially when compared to the pressure w.r.t. x.

Here, we notice that, the pressure inside the boundary layer at a given x is equivalent to the pressure outside the boundary layer at x. In our case, for an inviscid (potential) flow, the pressure is constant, p_∞. This removes the pressure gradient term.

With these, the solution to the two-system of equation,

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0 \\ \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) &= \mu\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

was given by Prandtl's student.

The stream function is given by $\Psi(x, y)$ where

$$\begin{aligned}u &= \frac{\partial \Psi}{\partial y} \\ v &= -\frac{\partial \Psi}{\partial x}\end{aligned}$$

which breaks down the problem further. Using the stream function, the continuity and x-momentum equation now becomes

$$\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial^2\Psi}{\partial x\partial y}=0$$
$$\frac{\partial\Psi}{\partial y}\frac{\partial^2\Psi}{\partial x\partial y}-\frac{\partial\Psi}{\partial x}\frac{\partial^2\Psi}{\partial y^2}=\nu\frac{\partial^3\Psi}{\partial y^3}$$

where $\nu=\mu/\rho$.

We can introduce a similarity variable to collapse all the $u(x,y)$ distribution, since they have identical profiles along x. Using non-dimensionalization, we write it as

$$\frac{u}{u_\infty}=F\left(\frac{y}{\delta(x)}\right)=F(\eta)$$

where η is our similarity variable.

Since $\Psi=\int u\,\mathrm{d}y=\int u_\infty F(\eta)\delta(x)\,\mathrm{d}\eta$, we now get a solution for the stream function:

$$\Psi=u_\infty\delta(x)f(\eta).$$

Plugging this into the x-momentum equation, we get

$$\frac{\partial\Psi}{\partial y}=u_\infty\frac{\mathrm{d}f}{\mathrm{d}\eta}$$

to be continued...

12 Lecture 12

Continuing from last time...

When the equation for Φ was plugged into the x momentum equation, the intermediate step was

$$-\frac{u_\infty}{\eta}\frac{\mathrm{d}\delta}{\mathrm{d}x}\delta(x)f(\eta)\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2}=\frac{\mathrm{d}^3f}{\mathrm{d}\eta^3}$$

where we realize that there is a component that is in terms of x; if we want to have this be a function of η only, then the term must be constant:

$$\text{constant}=\frac{u_\infty}{\eta}\frac{\mathrm{d}\delta}{\mathrm{d}x}\delta(x)$$

which is typically taken to be 1/2. Then, solving for δ, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\cancel{1}}{\cancel{2}}\delta^2\right)=\frac{\cancel{1}}{\cancel{2}}\frac{\eta}{u_\infty}$$
$$\Rightarrow\delta(x)=\sqrt{\frac{\eta x}{u_\infty}}.$$

Continuing with the initial equation, we have

$$2\frac{\mathrm{d}^3f}{\mathrm{d}\eta^3}+f\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2}=0$$

which is the Blasius equation. To solve this, we convert it to a system of first order equations. Letting

$$g=\frac{\mathrm{d}f}{\mathrm{d}\eta}$$
$$h=\frac{\mathrm{d}^2f}{\mathrm{d}\eta^2}$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\begin{bmatrix}f\\g\\h\end{bmatrix}=\begin{bmatrix}g\\h\\-\frac{1}{2}fh\end{bmatrix}$$

which can then be used in the Runge-Kutta algorithm for $\eta>0$.

We require knowing the initial conditions f,g,h. We do not directly know this, but can find them through other equations. First, using the equation for $u=\frac{\partial\Phi}{\partial y}$, we get

$$g=\frac{u}{u_\infty}.$$

Then, using $v=-\frac{\Phi}{x}$, we get

$$v=-\left[u_\infty\frac{\mathrm{d}\delta}{\mathrm{d}x}f+u_\infty\delta\frac{\mathrm{d}f}{\mathrm{d}\eta}\frac{\partial\eta}{\partial x}\right]$$
$$=-u_\infty\left[\frac{\mathrm{d}\delta}{\mathrm{d}x}f+\delta\frac{\mathrm{d}f}{\mathrm{d}\eta}\left(-\frac{y}{\delta^2}\right)\frac{\mathrm{d}\delta}{\mathrm{d}x}\right]$$
$$=-u_\infty\frac{\mathrm{d}\delta}{\mathrm{d}x}\left[f-\frac{\mathrm{d}f}{\mathrm{d}\eta}\eta\right].$$

Note that, our boundary conditions are:

- $u=0$ at $y=0$
- $v=0$ at $y=0$
- $u\to u_\infty$ as $y\to\infty$

The last condition is problematic, since Runge-Kutta wants the conditions at $y=0$, but the physical problem only offers 2. We are forced to *guess* h at $\eta=0$, such that our guess must lead to the following:

- $g=0$ at $\eta=0$
- $f=0$ at $\eta=0$
- $g\to1$ at $\eta\to\infty$, for a guessed h at $\eta=0$

The h at $\eta=0$ is known as the shooting parameter. The correct value is 0.332. Notice that $g=0.99$ at $\eta=4.92$ because of the equation

$$\frac{\delta}{\sqrt{\frac{\nu x}{u_\infty}}}=4.92.$$

We can re-write this as

$$\delta(x)=4.92\sqrt{\frac{\nu x}{u_\infty}}.\tag{101}$$

Notice that

$$\frac{\delta}{x}=4.92\mathrm{Re}_x^{-\frac{1}{2}}$$

and, consequently, this only applies to large Reynolds number only, and also the equation does not apply to the leading edge.

BLT is good for $\frac{\delta}{x}<\frac{1}{5}$ which translates to

$$\boxed{x>600\frac{\eta}{u_\infty}}$$

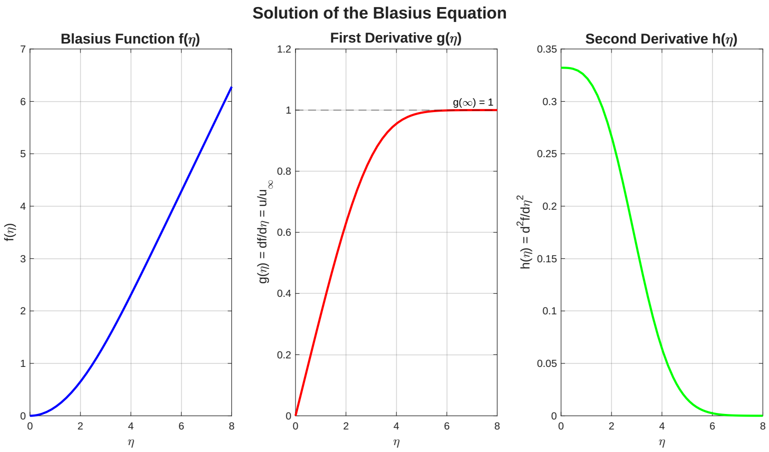

Figure 1: Numeric solutions to the Blasius equation for a flat plate

which is typically *very small*.

The non-zero slope of the $u(y)$ close to the surface is due to a shear stress. The shear stress at the surface is given as

$$\tau_w = \mu \left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{102}$$

which, more practically, is

$$\begin{aligned}\tau_w &= \mu \frac{\partial}{\partial y}\left(u_\infty \frac{\mathrm{d}f}{\mathrm{d}\eta}\right)_{y=0}\\ &= \mu u_\infty \left(\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\frac{\partial \eta}{\partial y}\right)_{\eta=0}\\ &= \mu u_\infty \left(\frac{\mathrm{d}^2 f}{\mathrm{d}\eta^2}\right)_{\eta=0}\sqrt{\frac{u_\infty}{\nu x}}\\ &= \boxed{0.332\mu u_\infty \sqrt{\frac{u_\infty}{\nu x}}}\end{aligned} \tag{103}$$

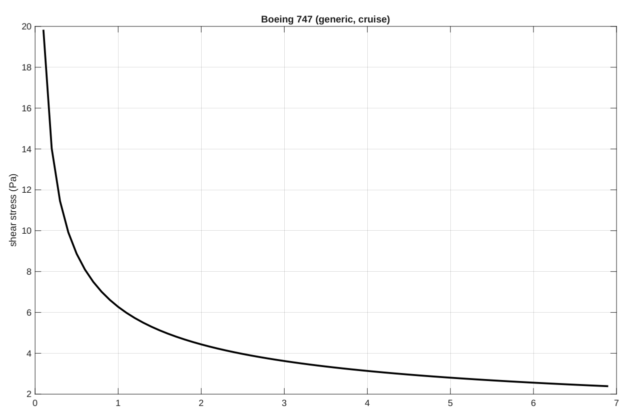

Figure 2: Shear stress on the wing of a Boeing 747
$\nu = 9.98 \times 10^{-6}, \mu = 15.1 \times 10^{-6}, u_\infty = 250, T = -40°C, AR = 8.6$

The skin friction coefficient is defined as

$$C_f \equiv \frac{\tau_w}{\frac{1}{2}\rho u_\infty^2} \tag{104}$$

which, practically, is

$$\boxed{C_f = 0.664\mathrm{Re}_x^{-1/2}}$$

A powerful tool in this case is integral methods. Here, the time rate of change of the x-momentum is equal to the net force in the x direction added to the change in the momentum w.r.t. time onto the control volume.

13 Lecture 13

Last class, we began working on the integral form of the equation. The streamline and boundary layer thickness are not the same. The continuity equation is

$$\int_0^\delta \rho u \, dy$$

on the left going to the right, and coming out on the left is

$$\int_0^\delta \rho u \, dy + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho u \, dy \, \mathrm{d}x$$

and is shown in figure 3.

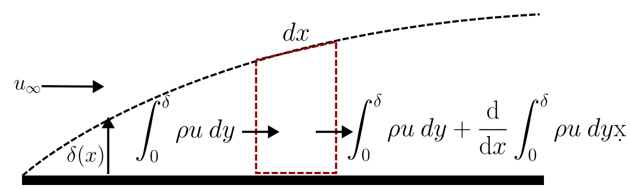

Figure 3: Control Volume equations for Integral, Continuity

The x-momentum equation is similarly

$$\int_0^\delta \rho u^2 \, dy \rightarrow \boxed{\text{control volume}} \rightarrow \int_0^\delta \rho u^2 \, dy + \frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho u^2 \, dy \, \mathrm{d}x.$$

Momentum gets carried into the control volume, which is represented by

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho u \, dy \, \mathrm{d}x u_\infty$$

Note that there is a shear stress going in the opposite stress, which stops the fluid:

$$\leftarrow \tau_w$$

The book derives this in a different way. The shear stress is given as

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho(u^2 - u u_\infty) \, dy = -\tau_w. \tag{105}$$

This is the x-momentum integral equation, shown in the book as 6.24.

From previous class, we know that

$$\frac{u}{u_\infty} = f\left(\frac{y}{\sqrt{\frac{\nu x}{u_\infty}}}\right) = f\left(\frac{y}{\delta}\right)$$

and combining this with the equation 105, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\int_0^1\left[\left(\frac{u}{u_\infty}\right)^2 u_\infty^2-\left(\frac{u}{u_\infty}\right)u_\infty^2\right]\mathrm{d}\left(\frac{y}{\delta}\right)\delta\right\} = -\frac{1}{\rho}\mu\left(\frac{\partial(u/u_\infty)u_\infty}{\partial(y/\delta)\delta}\right)_{y/\delta=0}.$$

Again substituting $u/u_\infty = f(y/\delta)$, we obtain a solution that is an oppproximate.

This assumption allows us to have a cubic polynomial, which allows a lot of fitting:

$$\frac{u}{u_\infty} = a + b\left(\frac{y}{\delta}\right) + c\left(\frac{y}{\delta}\right)^2 + d\left(\frac{y}{\delta}\right)^3$$

with the following boundary conditions:

- $u/u_\infty = 0$ at $y/\delta = 0$
- $u/u_\infty = 1$ at $y/\delta = 1$
- $\frac{\partial(u/u_\infty)}{\partial(y/\delta)} = 0$ at $y/\delta = 1$
- $\frac{\partial^2(u/u_\infty)}{\partial(y/\delta)^2} = 0$ at $y/\delta = 0$

The last one is coming from the fact that the vertical component in the following is zero:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \mu\frac{\partial^2 u}{\partial y^2}.$$

Doing this, we find the following:

$$\frac{u}{u_\infty} = \frac{3}{2}\left(\frac{y}{\delta}\right) - \frac{1}{2}\left(\frac{y}{\delta}\right)^3 \tag{106}$$

where $a = c = 0$. The third boundary condition is showcased in figure 4.

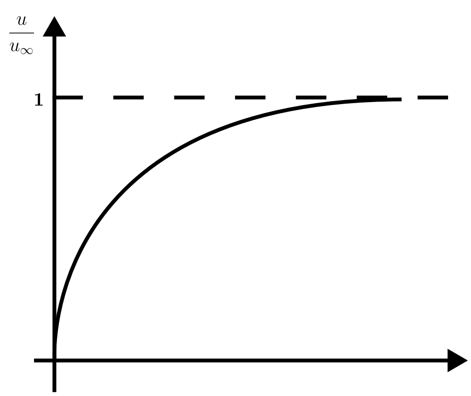

Figure 4: Third Boundary Condition

Substituting the approximation 106 into the earlier integral, we get the long equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\delta\int_0^1\left[\left(\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)^2-\left(\frac{3}{2}\left(\frac{y}{\delta}\right)-\frac{1}{2}\left(\frac{y}{\delta}\right)^3\right)\right]\mathrm{d}\left(\frac{y}{\delta}\right)\right\} = -\frac{\nu}{u_\infty\delta}\cdot\frac{3}{2}.$$

Although this is a large equation, this can be evaluated to a simple

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\delta\left(-\frac{39}{280}\right)\right\} = \frac{\nu}{u_\infty\delta}\frac{3}{2}$$

Solving this,

$$\delta^2 = \frac{\nu}{u_\infty}\frac{280}{13}x + \cancel{C_1}$$

yields the solution

$$\boxed{\delta(x) = 4.64\sqrt{\frac{\nu x}{u_\infty}}}. \tag{107}$$

Compare this equation 107 with the equation 101, we get 4.64 in contrast to 4.92, respectively.

Knowing this, we get

$$\frac{\delta}{x} = 4.64\sqrt{\frac{\nu}{u_\infty x}} = 4.64\mathrm{Re}_x^{-\frac{1}{2}}$$

which can be used to compute τ_w and C_f.

In the homework, we are asked to assume a linear function for u/u_∞:

$$\frac{u}{u_\infty} = \frac{y}{\delta}$$

instead of the cubic approximation.

The comparisons of solutions is shown in figure 5

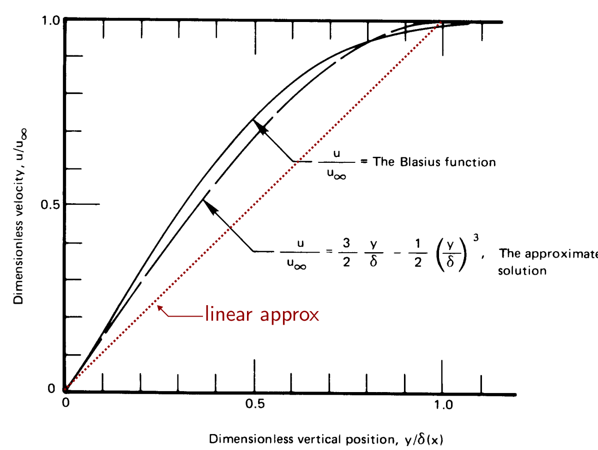

Figure 5: Comparison of linear, differential, and integral solutions

Integral equations are more forgiving: we can make poorer approximations, and still get good results. Finite element methods utilize integral methods. Piece-wise linear approximations are performed.

The thermal boundary layer is different. To find the temperature in the boundary layer, we have to bring back the energy equation. The thermal boundary layer depends on the energy equations, given by

$$\rho c_p\frac{\mathrm{D}T}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} + k\nabla^2 T + \overset{\text{neglect}}{\cancel{\Phi}}$$

In steady state two-dimensional flow, this translates to

$$\rho c_p\left(u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}\right)=\cancelto{\text{flat plate}}{\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)}$$
$$+k\left(\cancel{\frac{\partial^2 T}{\partial x^2}}+\frac{\partial^2 T}{\partial y^2}\right)$$

In the above, the rate of change in temperature along the x direction is small. Setting

$$\Theta=\frac{T-T_w}{T_\infty-T_w}$$

we get

$$\rho c_p\left(u\frac{\partial\Theta}{\partial x}\cancel{(T_\infty-T_w)}+v\frac{\partial\Theta}{\partial y}\cancel{(T_\infty-T_w)}\right)=k\left(\frac{\partial^2\Theta}{\partial y^2}\cancel{(T_\infty-T_w)}\right)$$

where the cancelled terms cancel because they are constants. This simplifies to

$$u\frac{\partial\Theta}{\partial x}+v\frac{\partial\Theta}{\partial y}=\alpha\frac{\partial^2\Theta}{\partial y^2}.$$

Now, when we bring in the momentum equation in x-direction , we realize that they are very similar:

$$u\frac{\partial(u/u_\infty)}{\partial x}+v\frac{\partial(u/u_\infty)}{\partial y}=\nu\frac{\partial^2(u/u_\infty)}{\partial y^2}$$

If $\nu=\alpha$, then the differential equation for u/u_∞ is same as the differential equation for Θ , and the boundary conditions are the same. This is definitely possible, since both ν and α have the same dimensions. Boundary conditions also can be the same.

More formally, if $\nu=\alpha$, then

$$\boxed{\frac{u}{u_\infty}=\Theta=f'(\eta)} \tag{108}$$

where f is the Blasius function, where

$$\eta=\frac{y}{\sqrt{\frac{\nu x}{u_\infty}}}.$$

In other words, by knowing properties of the fluid problem, we can understand the thermal problem.

When $\nu=\alpha$,

$$\frac{T-T_w}{T_\infty-T_w}=f'(\eta)$$
$$\Rightarrow T=T_w+(T_\infty-T_w)f'(\eta).$$

There is a heat flux q_w going from the flat plat to the fluid, where we assume that the flat plate is hot. We know how to compute this using Fourier's law, since

$$q_w=-k_f\left(\frac{\partial T}{\partial y}\right)_{y=0}$$

and taking the derivative, we get

$$q_w=-k_f(T_\infty-T_w)\left(f''(\eta)\frac{\partial\eta}{\partial y}\right)_{\eta=0}$$

which is the wall heat flux:

$$q_w=\boxed{k_f(T_w-T_\infty)\left(\frac{f''(\eta)}{\sqrt{\frac{\eta x}{u_\infty}}}\right)_{\eta=0}} \tag{109}$$

Using the numerical solution that we got in the earlier 'shooting problem', a numeric solution is

$$q_w=k_f(T_w-T_\infty)(0.332)\sqrt{\frac{u_\infty}{\eta x}} \tag{110}$$

We can now define h, where

$$q_w=h(T_w-T_\infty)$$

as

$$h=k\sqrt{\frac{u_\infty}{\nu x}}\cdot 0.332 \tag{111}$$

which is no longer a simple $\bar{h}$ average. This is Nusselt number, defined as

$$\boxed{\mathrm{Nu}_x\equiv\frac{hx}{k}} \tag{112}$$

and numerically results in

$$\mathrm{Nu}_x=0.332\sqrt{\frac{u_\infty x}{v}}=0.332\cdot\mathrm{Re}_x^{1/2} \tag{113}$$

but is *only valid if* $\nu=\alpha$.

Example 6.4 from the book is described in figure 6. This is for

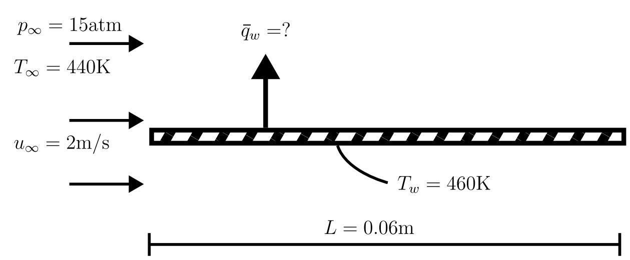

Figure 6: Example 6.4

water, and as a result, the $\nu=\alpha=1.724\times10^{-7}\,\mathrm{m}^2/\mathrm{s}$. Solving this, we have

$$\bar{q}_w=\frac{1}{L}\int_0^L q_w\,\mathrm{d}x$$
$$=\frac{1}{L}k(T_w-T_\infty)(0.332)\sqrt{\frac{\nu}{u_\infty}}\int_0^L\frac{1}{\sqrt{x}}\,\mathrm{d}x$$

Note that there is a singularity at $x=0$, but it is an integrable singularity. This is caused by the breakdown of boundary layer theory at the tip. The solution is showcased in figure 7. Note that the Prandtl number is given by $\mathrm{Pr}=\frac{\nu}{\alpha}$ and when this is 1, this solution method can be used. This is realistic for certain gasses.

- Diatomic gasses such as nitrogen and oxygen gasses, it is close to 5/7

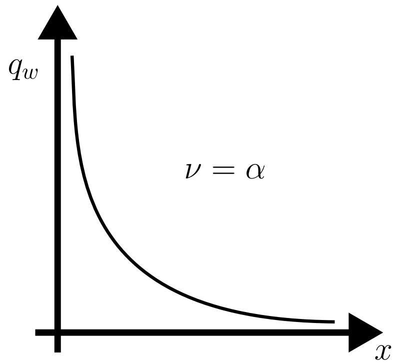

Figure 7: Example 6.4 solution

- For liquid metals, α is large, so the number is very small.
- For very complex large fluids, ν is large, so the Prandtl number is very large.
- Liquids have higher sensitivity of Prandtl number than gasses; it generally decreases as temperature rises: $\mathrm{Pr} \propto 1/T$.

. When the Prandtl number is not 1, the difference in boundary laer exists and is showcased in figure 8

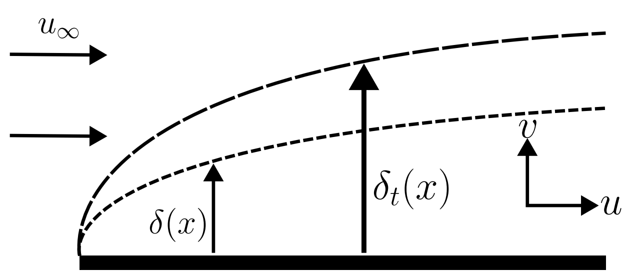

Figure 8: Difference of boundary layers

14 Lecture 14

In our flat plate problem, there is a fluid mechanics problem consisting of the viscous boundary layer, $\delta(x)$, and a thermal boundary layer, $\delta_t(x)$. In *general*, $\delta(x) \neq \delta_t(x)$; only under the special circumstance of $\nu = \alpha$ does it equate to each other.

In the thermal boundary layer, the equation for temperature was given as

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \frac{k}{\rho c_p}\left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{\partial^2 T}{\partial y^2}\right) + \cancel{\frac{1}{\rho c_p}\Phi}$$

where either the similarity solution or the energy integral equation can be used. Figure 9 is similar to the previous figure 3, but this involves thermal boundary layer. Note that

$$\hat{h} - \hat{h}_\infty \approx c_p(T - T_\infty)$$

and

$$q_w = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta_t} \rho u(\hat{h} - \hat{h}_\infty)\,\mathrm{d}y$$

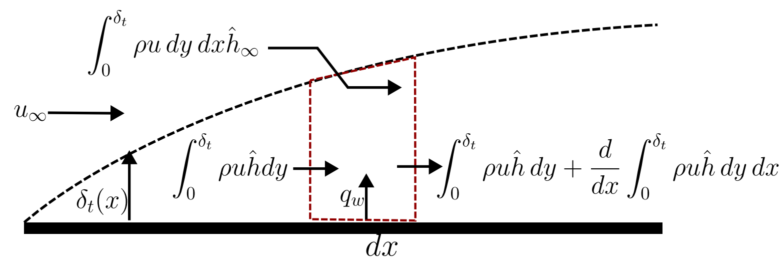

Figure 9: Control Volume for Integral Thermal BLT

and the solution to the differential equation is slightly more difficult: In this analysis, we set

$$\Theta = \frac{T - T_\infty}{T_w - T_\infty}$$

where we also say that Θ is a function of $\eta = y/\delta_t$. Applying this on:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^{\delta_t} u(T - T_\infty)\,\mathrm{d}y = -\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0},$$

we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \left(\frac{u}{u_\infty}\right) u_\infty \Theta \cancel{(T_w - T_\infty)} \delta_t\,\mathrm{d}\eta$$
$$= -\alpha \frac{\partial}{\partial \eta}(\cancel{T_\infty} + \Theta\cancel{(T_w - T_\infty)})\frac{1}{\delta_t}\Big|_{\eta=0}.$$

The cancellations arrive from taking the derivative of constants, and removing terms that exist on both sides. Evaluating this,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\delta_t \int_0^1 \left(\frac{u}{u_\infty}\right)\Theta\,\mathrm{d}\eta\right\} = -\frac{\alpha}{u_\infty}\frac{1}{\delta_t}\left(\frac{\partial \Theta}{\partial \eta}\right)_{\eta=0}$$

We will assume that

$$\frac{u}{u_\infty} = \frac{3}{2}\frac{y}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3$$

which was obtained last time, but this is δ , not δ_t. Consequently, we modify this as

$$\begin{aligned}\frac{u}{u_\infty} &= \frac{3}{2}\frac{y}{\delta_t}\frac{\delta_t}{\delta} - \frac{1}{2}\left(\frac{y}{\delta}\right)^3\left(\frac{\delta_t}{\delta}\right)^3 \\ &= \frac{3}{2}\eta\phi - \frac{1}{2}\eta^3\phi^3\end{aligned}$$

where $\phi \equiv \delta_t/\delta$. Looking at figure 8, notice that the above equation is only valid for

$$\delta > \delta_t \implies \mathrm{Pr} > 1 \tag{114}$$

since $\mathrm{Pr} = \nu/\alpha$.

Assume that

$$\Theta = a + b\eta + c\eta^2 + d\eta^3$$

we get the boundary conditions

1. $\Theta = 1$ at $\eta = 0$
2. $\Theta = 0$ at $\eta = 1$

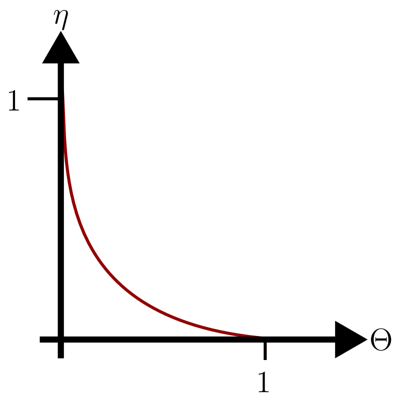

Figure 10: Boundary Conditions for Θ w.r.t. η

3. $\frac{\partial \Theta}{\partial \eta} = 0$ at $\eta = 1$
4. $\frac{\partial^2 \Theta}{\partial \eta^2} = 0$ at $\eta = 0$, this comes from the energy equation.

The conditions are shown in figure 10. The textbook solves for the constants a, b, c, d, which then yields

$$\Theta = 1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3.$$

Substituting this in to the earlier differential equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\delta_t \int_0^1 \left[\frac{3}{2}\eta\phi - \frac{1}{2}\eta\phi^3\right]\left[1 - \frac{3}{2}\eta + \frac{1}{2}\eta^3\right]\mathrm{d}\eta\right\} = -\frac{\alpha}{u_\infty}\frac{1}{\delta_t}\left(-\frac{3}{2}\right)$$

which then simplifies to

$$\delta_t \frac{\mathrm{d}}{\mathrm{d}x}\left\{\delta_t\left[\frac{3}{20}\phi - \frac{3}{280}\phi^3\right]\right\} = \frac{3}{2}\frac{\alpha}{u_\infty},$$

and again,

$$\frac{\cancel{1}}{\cancel{2}}\frac{\mathrm{d}}{\mathrm{d}x}\delta_t^2 = \frac{3}{\cancel{2}}\frac{\alpha}{u_\infty}\frac{1}{\frac{3}{2}\phi\left(1 - \frac{\phi^2}{14}\right)}$$

where we can take the integral. Note that the constant term cancels because we are already at the boundary condition:

$$\delta_t^2 = \frac{3\alpha}{u_\infty}\frac{x}{\frac{3}{20}\phi\left(1 - \frac{\phi^2}{14}\right)} + \cancel{C}$$

We now bring back the δ^2 term on the denominator:

$$\frac{\delta_t^2}{\delta^2} = \frac{3\alpha}{u_\infty}\frac{1}{\frac{3}{20}\phi\left(1 - \frac{\phi^2}{14}\right)}\frac{x}{\delta^2}.$$

Doing this is useful because, from the viscous boundary layer equation 107, the solution being plugged in yields

$$\frac{\delta_t^2}{\delta^2} = \frac{\cancel{3}\alpha}{\cancel{u_\infty}}\frac{1}{\frac{\cancel{3}}{20}\phi\left(1 - \frac{\phi^2}{14}\right)}\frac{\cancel{x}\cancel{u_\infty}}{(4.64)^2\nu\cancel{x}}$$

Note that it is crucial for the x to cancel out, otherwise our initial assumption would be wrong, that ϕ does not depend on x. This creates the relationship between the two boundary layers and the Prandtl number:

$$\frac{\delta_t^2}{\delta^2} = \frac{1}{\mathrm{Pr}}\frac{20}{(4.64)^2}\frac{1}{\phi\left(1 - \frac{\phi^2}{14}\right)} \tag{115}$$

The relation between ϕ and Pr is plotted in figure 11. We can

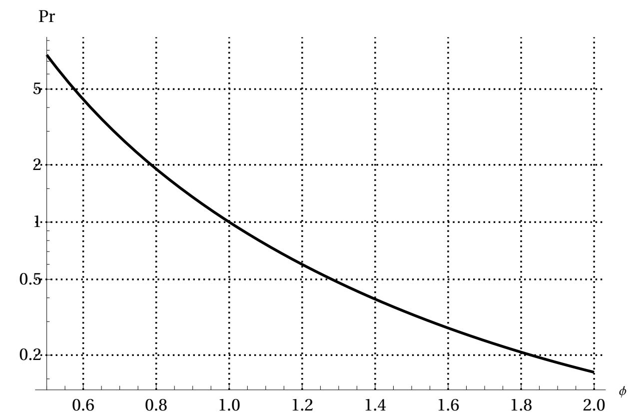

Figure 11: Relation between Pr and ϕ

approximate ϕ as

$$\phi = 0.967\mathrm{Pr}^{-1/3} \approx \mathrm{Pr}^{-1/3} \tag{116}$$

for $0.6 < \mathrm{Pr} < 50$.

Now, we solve for q_w:

$$\begin{aligned} q_w &= -k\frac{\partial}{\partial \eta}\left(\cancel{T_\infty} + (T_w - T_\infty)\Theta\right)\frac{1}{\delta_t}\Big|_{\eta=0} \\ &= -k(T_w - T_\infty)\left(\frac{\partial \Theta}{\partial \eta}\right)_{\eta=0}\frac{1}{\delta_t} \\ &= k(T_w - T_\infty)\left(\frac{3}{2}\right)\frac{\delta}{\delta_t}\frac{1}{\delta} \end{aligned}$$

Notice that δ/δ_t can be re-written as the Pr number:

$$q_w = k(T_w - T_\infty)\left(\frac{3}{2}\right)\mathrm{Pr}^{3/2}\frac{1}{\delta}$$

and we define h in $q_w = h(T_w - T_\infty)$ as

$$h = k\left(\frac{3}{2}\right)\mathrm{Pr}^{1/3}\frac{1}{\delta}$$

where we can now get another relation with the Nusselt number:

$$\mathrm{Nu}_x = \frac{hx}{k} = \frac{3}{2}\mathrm{Pr}^{1/3}\frac{x}{\delta}$$

Again, using equation 107, we now get an approximate solution for the Nusselt:

$$\boxed{\mathrm{Nu}_x \approx 0.331\mathrm{Pr}^{1/3}\mathrm{Re}_x^{1/2}}. \tag{117}$$

where

$$\text{Re}_x = \frac{u_\infty x}{\nu}.$$

Notice, again, that we are no longer using an 'average' $\bar{h}$. Liquid metals are great for transferring heat, due to their high heat densities. For these, Pr << 1. In this case, the thermal boundary layer is much larger: the u_∞ exists within the thermal boundary layer for a large region. This is shown in figure 12. The Péclet number

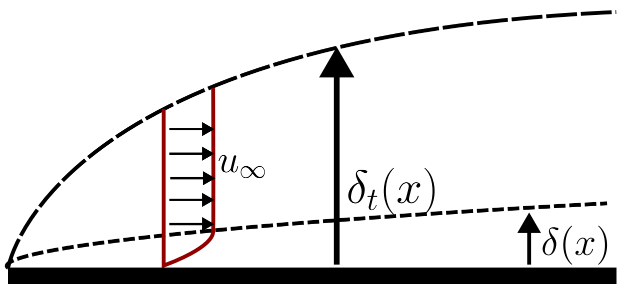

Figure 12: Boundary layer thicknesses for very small Pr

is given as

$$\text{Pe} = \frac{u_\infty x}{\alpha} \tag{118}$$

and indicates the ratio of fluid flow to heat transfer rate. When Pe ≫ 1, fluid's ability to absorb heat dominates, while Pe ≪ 1 effect of fluid's ability to transfer heat dominates.

In the differential equation, keep in mind that the largest term is $\frac{\partial u}{\partial y}$. The order of magnitude estimate is

$$\mu\left(\frac{\partial u}{\partial y}\right)^2 \approx \mu\left(\frac{u_\infty}{\delta}\right)^2$$

and applying this on

$$k\frac{\partial^2 T}{\partial y^2} \approx \frac{k(T_w - T_\infty)}{\delta^2}$$

we can get

$$\begin{aligned}\frac{\mu\left(\frac{\partial u}{\partial y}\right)^2}{k\frac{\partial^2 T}{\partial y^2}} &\approx \frac{\mu u_\infty^2}{\cancel{\delta^2}}\frac{\cancel{\delta^2}}{k(T_w - T_\infty)} \\ &\approx \frac{u_\infty^2}{\alpha\rho c_p}\frac{\nu\rho}{(T_w - T_\infty)} \\ &\approx \frac{\nu}{\alpha}\frac{u_\infty^2}{c_p(T_w - T_\infty)} \\ &\approx \frac{\nu}{\alpha}\cdot \text{Ec}\end{aligned}$$

where $\alpha = \frac{k}{\rho c_p}$ was substituted here. These yield

$$\frac{\mu\left(\frac{\partial u}{\partial y}\right)^2}{k\frac{\partial^2 T}{\partial y^2}} \approx \text{Pr}\cdot\text{Ec} \tag{119}$$

where Ec is the Eckert number, and when the product of the two is small, the viscous dissipation is not important. When this is large, such as 1 or higher, we can no longer neglect viscous dissipation. Reynolds number is not important when the BLT is small. Peclet number becomes important.

15 Lecture 15

Homework 7 is due this Friday.

If we can identify the temperature distribution in the fluid, we can identify the heat flux from the plate to the fluid. Last class, we identified an analytical solution. This involved equation 117. This could then be used in

$$q_w(x) = h(x)(T_w - T_\infty).$$

In the homework, we'll be asked to find a solution that is already in the book, but using the differential method.

A different problem is the 'unheated starting length', shown in figure 13, has a solution of:

$$\text{Nu}_x = \frac{0.332\text{Pr}^{1/3}\text{Re}_x^{1/2}}{\left[1-(x_0/x)^{3/4}\right]^{1/3}} \tag{120}$$

for $x > x_0$.

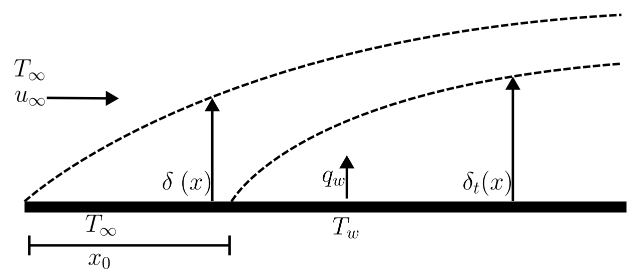

Figure 13: Unheated starting length

In a uniform heat flux problem, shown in 14, has the solution of

$$\text{Nu}_x = 0.453\text{Pr}^{1/3}\text{Re}_x^{1/2} \tag{121}$$

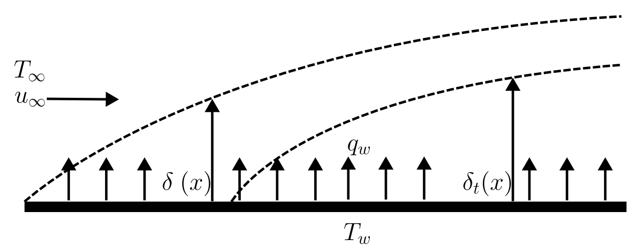

Figure 14: Uniform heat flux problem

We assume steady laminar flow in our solutions, and as a result, our solutions do not hold true when turbulence begins. In turbulence, there is unsteady vortex formation. We will not go into in-depth analysis of turbulent flow, but we will starting doing a small dive into it. In real common flows, flow becomes turbulent very quickly. Analysis is based on a lot of measurements. Velocity of fluid at a particular point as a function of time is done through several techniques such as hot wire techniques and laser.

There is irregular unsteady velocity over time at a particular point in turbulent flow, but there exists a mean flow. The average flow and fluctuation can be separated. This is known as the time average.

$$u = \bar{u} + u'$$

where

$$\hat{u'} = \frac{1}{T}\int_0^T u'(t)\,\mathrm{d}t = 0 \tag{122}$$

is shown in 15. This involves using continuity equation, and as-

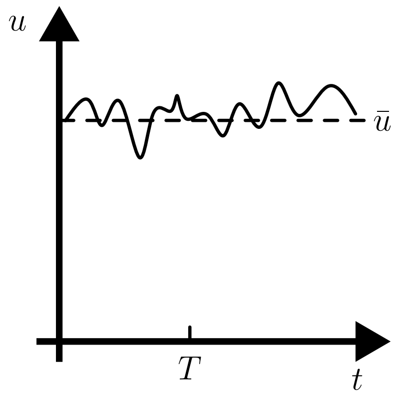

Figure 15: Decomposition of turbulent flow at a *point*

suming incompressible flow. In such a flow, in 2D, we have our typical continuity equation

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

but an unsteady x-momentum equation given as

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \frac{\partial^2 u}{\partial y^2}\right). \tag{123}$$

Notice how the differential equations are still valid in turbulent flow. Also, notice that the time derivative is present.

Then, we set the following:

$$u = \bar{u} + u' \tag{124}$$
$$v = \bar{v} + v' \tag{125}$$
$$p = \bar{p} + p' \tag{126}$$

which transforms the continuity equation:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial v'}{\partial y} = 0 \tag{127}$$

We should be able to average each term, in which case, we should be able to take out the x derivative, since there is an integral of time. For example, for the first term, taking the average of an average is the original average itself:

$$\overline{\frac{\partial \bar{u}}{\partial x}} = \frac{\partial}{\partial x}\overline{(\bar{u})} = \frac{\partial \bar{u}}{\partial x}$$

whereas the fluctuation yields zero:

$$\frac{\partial}{\partial x}\overline{u'} = 0$$

which now transforms the continuity equation into

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

which is a good result because this states that the average flow also satisfies the continuity equation. This implies that equation 127 now also means

$$\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0.$$

Next, we deal with the x-momentum equation 123. Substituting in the decomposition yields

$$\rho\left(\left(\frac{\partial \bar{u}}{\partial t} + \frac{\partial u'}{\partial t}\right) + (\bar{u} + u')\left(\frac{\partial \bar{u}}{\partial x} + \frac{\partial u'}{\partial x}\right) + (\bar{v} + v')\left(\frac{\partial \bar{u}}{\partial y} + \frac{\partial u'}{\partial y}\right)\right)$$
$$= -\left(\frac{\partial \bar{p}}{\partial x} + \frac{\partial p'}{\partial x}\right) + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}} + \left(\frac{\partial^2 \bar{u}}{\partial y^2} + \frac{\partial^2 u'}{\partial y^2}\right)\right). \tag{128}$$

Two random quantities multiplied and averaged doesn't lead to a zero, rather, produce a correlation; hence we must keep the term. Otherwise, a simple average of a regular random variable does lead to zero. This now leads to

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \overline{u'\frac{\partial u'}{\partial x}} + \bar{v}\frac{\partial \bar{u}}{\partial y} + \overline{v'\frac{\partial u'}{\partial y}}\right) = -\frac{\partial \bar{p}}{\partial x} + \mu\frac{\partial^2 \bar{u}}{\partial y^2}$$

Notice that, all the terms of the overline indicate the property of the fluctuation of turbulence. In other words, the turbulence complication still appears. Certain identities appear here:

$$u'\frac{\partial u'}{\partial x} = \frac{\partial}{\partial x}\left(u'^2\right) - u'\frac{\partial u'}{\partial x}$$
$$v'\frac{\partial v'}{\partial y} = \frac{\partial}{\partial y}\left(u'v'\right) - u'\frac{\partial v'}{\partial y}$$

and we can take the average off all the terms:

$$\overline{u'\frac{\partial u'}{\partial x}} = \overline{\frac{\partial}{\partial x}\left(u'^2\right)} - \overline{u'\frac{\partial u'}{\partial x}}$$
$$\overline{v'\frac{\partial v'}{\partial y}} = \overline{\frac{\partial}{\partial y}\left(u'v'\right)} - \overline{u'\frac{\partial v'}{\partial y}}$$

Notice that the terms with squares no longer yield zero. That is the average of the square. Adding the two equations together and taking another average yields

$$\overline{u'\frac{\partial u'}{\partial x}} + \overline{v'\frac{\partial u'}{\partial y}} = \frac{\partial}{\partial x}\left(\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\overline{u'v'}\right) - \cancel{\overline{u'\left(\frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y}\right)}}$$

Plugging this into the earlier equation and simplifying:

$$\rho\left(\frac{\partial \bar{u}}{\partial t} + \bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right)$$
$$= -\frac{\partial}{\partial x}\left(\bar{p} + \underbrace{\rho\overline{u'^2}}_{\text{Re's pressure}}\right)$$
$$+ \frac{\partial}{\partial y}\left(\underbrace{\mu\frac{\partial \bar{u}}{\partial y}}_{\bar{\tau}} - \underbrace{\rho\overline{u'v'}}_{\text{Re's shear stress}}\right)$$

Letting

$$-\overline{u'v'} = \epsilon_m\frac{\partial \bar{u}}{\partial y}$$

with ϵ_m being the 'eddy viscosity', yields

$$\rho\frac{\mathrm{D}\bar{u}}{\mathrm{D}t} = -\frac{\partial}{\partial x}\left(\bar{p} + \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left(\mu\frac{\partial\bar{u}}{\partial y} + \rho\epsilon_m\frac{\partial\bar{u}}{\partial y}\right)$$

which is the right hand side of our earlier equation. Then, dividing by ρ:

$$\frac{\mathrm{D}\bar{u}}{\mathrm{D}t} = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(\bar{p} + \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left((\nu + \epsilon_m)\frac{\partial\bar{u}}{\partial y}\right)$$

where $(\nu + \epsilon_m)$ is the combination of the true viscosity and eddy viscosity. Note that ν is a *fluid* property, whereas ϵ_m is a *flow* property. No one yet knows how to bring this back to the original flow equation.

The mixing length theory/hypothesis was developed in the context of boundary layer theory. Figure 16 shows the mixing length of the eddies. The perturbation V' changes the flow. Prandtl concluded that the average of the fluctuating term is approximately

$$-\overline{u'v'} \approx \ell^2\left(\frac{\partial\bar{u}}{\partial y}\right)^2.$$

When we are talking about fluid in laminar flow, they are flowing

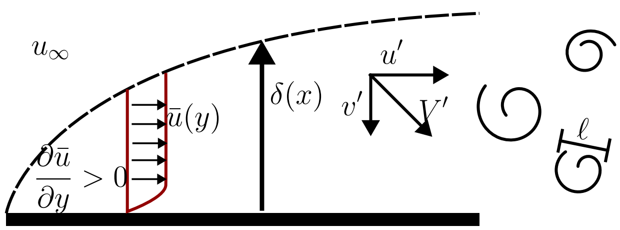

Figure 16: Maximum length problem

in a straight line at the macroscopic scale. However, at the microscopic scale, there is a random motion. Statistical theory of fluid mechanics combines the macroscopic motion and the microscopic motion. Prandtl includes some of the random motions (denoted by $\square'$) in the macroscopic motion. The slow molecules induce drag on the fast molecules, and the fast molecules try to speed up the slow molecules.

In all cases,

$$-\overline{u'v'} = \ell^2\left|\frac{\partial\bar{u}}{\partial y}\right|\frac{\partial\bar{u}}{\partial y}$$

and combining with

$$-\overline{u'v'} = \epsilon_m\frac{\partial\bar{u}}{\partial y}$$

we get

$$\epsilon_m = \ell^2\left|\frac{\partial\bar{u}}{\partial y}\right|. \tag{129}$$

We now get a new x-momentum equation:

$$\left(\frac{\partial\bar{u}}{\partial t} + \bar{u}\frac{\partial\bar{u}}{\partial x} + \bar{v}\frac{\partial\bar{u}}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial}{\partial x}\left(\bar{p} + \rho\overline{u'^2}\right) + \frac{\partial}{\partial y}\left((\nu + \ell^2\left|\frac{\partial\bar{u}}{\partial y}\right|)\frac{\partial\bar{u}}{\partial y}\right) \tag{130}$$

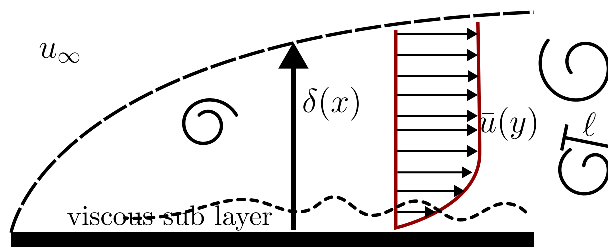

Figure 17: Viscous sub layer and eddy viscosity

The most important terms to note here is the $\rho\overline{u'^2}$ and the ϵ_m term that got substituted.

Close to the wall, the viscosity or shear stress dominates; there's a viscous sub layer shown in Once we get the mean velocity profile, we can bring it to the momentum integral equation. The momentum integral equation would yield a boundary layer thickness and a shear stress at the wall.

The skin friction coefficient in the turbulent region, with a 1–2% accuracy, is

$$C_f(x) = \frac{0.455}{\left[\ln(0.06\mathrm{Re}_x)\right]^2}. \tag{131}$$

All these were the fluid flow problem, not a thermal problem. To deal with the thermal problem, we introduce the energy equation like before. In the boundary layer, the energy equation has a convective term for the temperature:

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\left(\frac{\partial^2 T}{\partial y^2}\right)$$

With temperature, we introduce the decomposition equations 124 and 125, and:

$$T = \bar{T} + T'$$

which leads to

$$\rho c_p\left(\frac{\partial\bar{T}}{\partial t} + \bar{u}\frac{\partial\bar{T}}{\partial x} + \bar{v}\frac{\partial\bar{T}}{\partial y}\right) = \frac{\partial}{\partial y}\left(k\frac{\partial\bar{T}}{\partial y} - \rho c_p\overline{u'T'}\right)$$

where the last term indicates the heat flux due to turbulent mixing.

16 Lecture 16

Nearly the entirety of chapter 6 is about flow over a flat plate. The unsteady form of the energy equation is

$$\rho c_p\left(\frac{\partial T}{\partial t} + u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = k\frac{\partial^2 T}{\partial y^2}$$

and we neglect conduction in the axial direction, which is valid when the Eckert number is small. For turbulent flow, we decompose the flow in to two terms: $u = \bar{u} + u'$ as done last time. Taking the averages, the energy equation becomes

$$\rho c_p \left(\frac{\partial \bar{T}}{\partial t} + \bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} \right) = \frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} - \rho c_p \overline{u'T'} \right)$$

where the last term is interpreted as an additional heat flux due to the turbulent fluctuations. The turbulent fluctuations also add to the shear stress. We defined

$$\epsilon_h \frac{\partial \bar{T}}{\partial y} = -\overline{u'T'}$$

and ϵ_h is the 'eddy diffusivity of heat'. Notice that this is slightly different from ϵ_m.

We can assume that the mean temperature w.r.t. time is steady (i.e., the derivative is zero), which then gets us

$$\rho c_p \left(\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} \right) = \frac{\partial}{\partial y}\left(k\frac{\partial \bar{T}}{\partial y} + \rho c_p \epsilon_h \frac{\partial \bar{T}}{\partial y} \right)$$

Dividing both sides by ρc_p, we get

$$\bar{u}\frac{\partial \bar{T}}{\partial x} + \bar{v}\frac{\partial \bar{T}}{\partial y} = \frac{\partial}{\partial y}\left((\alpha + \epsilon_h)\frac{\partial \bar{T}}{\partial y} \right)$$

This is very much like the laminar case; the ϵ_h is a flow property, which is problematic. In this equation, ϵ_h can be considered the effective thermal diffusivity due to turbulent fluctuations.

We need to describe ϵ_h somehow. The Prandtl number is a fluid property, since both ν/α do not depend on the flow. We define a *turbulent* Prandtl number which is described in terms of eddy diffusivity of heat:

$$\mathrm{Pr}_t = \frac{\epsilon_m}{\epsilon_h} \tag{132}$$

This is not true, but is close enough. The mechanism of eddy diffusivity of momentum and the eddy diffusivity of heat is same: turbulent mixing. Since this is in the same flow, and both are effect of the same thing, we can assume that

$$\mathrm{Pr}_t \approx 1 \implies \epsilon_h \approx \epsilon_m$$

in most cases. This enables us to find the thermal boundary layer and mean heat transfer. Figure 6.19 in the book showcases the accuracy of this. In laminar case, constant T_w or constant q_w distinction matters, but for turbulent case, it does not. The effects in the fluid matter more than the wall-fluid interface in turbulence. The book offers a transition equation from laminar to turbulent for q_w constant. For constant T_w, the fit is harder.

Chapter 7 discusses pipe flow, which is important since most fluids are transferred through pipes. In these problems, radial coordinates will be used, with the assumption that the flow is axis-symmetric.

Velocity in the x direction is given as $u(x, r, t)$, where, in steady flow, there is no time dependence. The surface of the pipe has a temperature of $T_w(x, t)$. The typical question in these problems is to find q_w. We can take Fourier's law for this. The wall surface is located at $r = R$.

In the flat plate problem, we defined

$$q_w = h(T_w - T_\infty)$$

but in the pipe problem, there is no 'infinity' for T_∞. Hence, we define

$$q_w = h(T_w - T_b) \tag{133}$$

where the T_b is the 'bulk' temperature. This is defined at a particular x. It is defined by

$$\dot{m}c_p T_b = \int_0^R \rho u c_p T \,\mathrm{d}A$$

$$\Rightarrow T_b = \frac{\int_0^R \rho u c_p T \,\mathrm{d}A}{\dot{m}c_p} \tag{134}$$

and is sometimes known as the 'bowl temperature'. If we instantly cut a pipe, and let a small finite time amount of the fluid fill a bowl, and let the bowl temperature equilibriate, we'd get the 'bowl temperature'. The bowl temperature is more realistic than the temperature distribution.

We'll neglect axial conduction, because it is much smaller than that of the radial direction. Figure 18 show cases the control volume. The differential equation we end up getting

$$\frac{\mathrm{d}T_b}{\mathrm{d}x} = \frac{q_w 2\pi R}{\dot{m}c_p} \tag{135}$$

when we consider that this is axis-symmetric. We typically assume that the pipe fluid starts with a uniform flow of u_∞, and over time, it becomes non-uniform: all fluid in the pipe becomes affected by viscosity. The boundary layers at all the 'differential flat plates' combine and create a velocity profile.

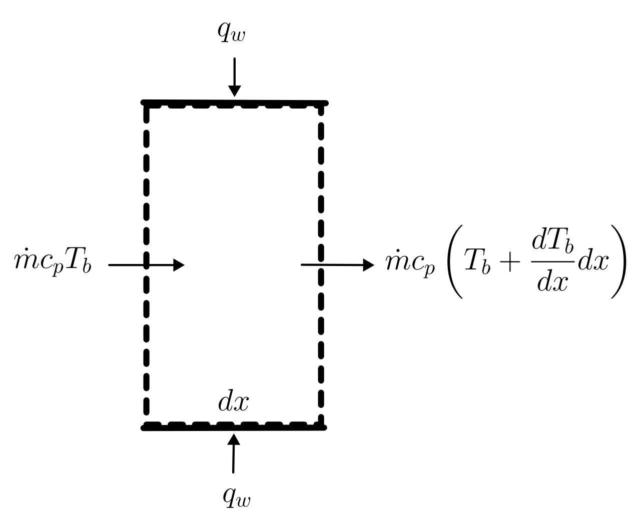

Figure 18: Pipe control volume

x_e is the 'entrance length' that the fluid has to go through to become fully viscous. This is shown in 19 and defined as

$$\frac{x_e}{D} \approx 0.03\mathrm{Re}_D \tag{136}$$

and the Reynolds number is given as

$$\mathrm{Re}_D = \frac{u_{avg} D}{\nu} \tag{137}$$

and the mass flow is given as

$$\dot{m} = \int \rho u \, \mathrm{d}A = \int_0^R \rho u(r) 2\pi R \, \mathrm{d}R \tag{138}$$

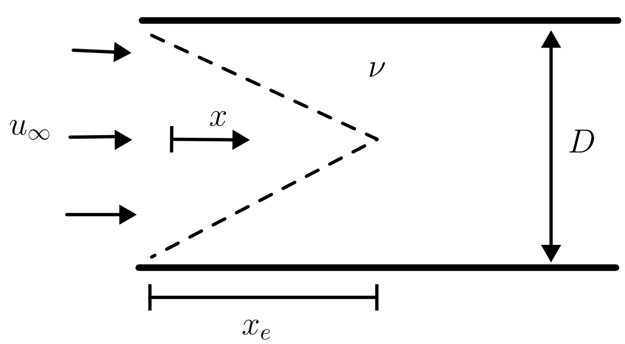

Figure 19: Entrance length

The average velocity is given as

$$u_{\mathrm{avg}} = \frac{\dot{m}}{\rho A} = -\frac{\mathrm{d}p}{\mathrm{d}x}\frac{R^2}{8\mu} \tag{139}$$

and the velocity profile in the fully developed region is

$$u(r) = -\frac{\mathrm{d}p}{\mathrm{d}x}\frac{R^2}{4\mu}\left[1 - \frac{r^2}{R^2}\right] \tag{140}$$

yielding

$$u(r) = 2u_{\mathrm{avg}}\left[1 - \frac{r^2}{R^2}\right] \tag{141}$$

with the maximum being

$$u_{\mathrm{max}} = -\frac{\mathrm{d}p}{\mathrm{d}x}\frac{R^2}{4\mu}. \tag{142}$$

shown in 20.

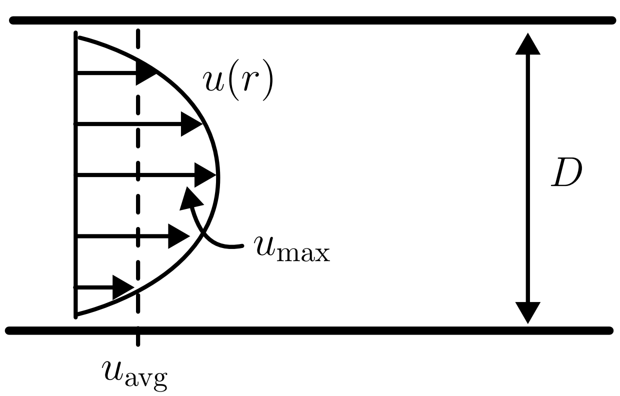

Figure 20: Pipe velocity profile

Like in the flat plate problem, the thermal flow will be slightly different. Typically, the wall temperature can be constant, or the heat flux can be constant. The 'constant heat flux' is realistic, such as the 'tankless water heater' devices (electric). Older homes use gas heating to heat up a reserve, and that is *not* constant heat flux. There is a drop in temperature in the middle of the pipe, and the wall temperature is higher. This is shown in figure 7.2 of the book, added here as 21, for the entrance region.

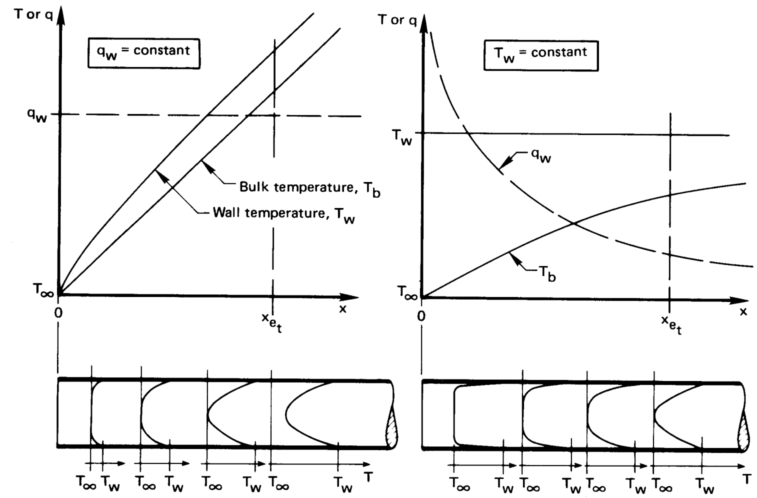

Figure 21: Temperature profile for constant heat flux and constant wall temperature in entrance region

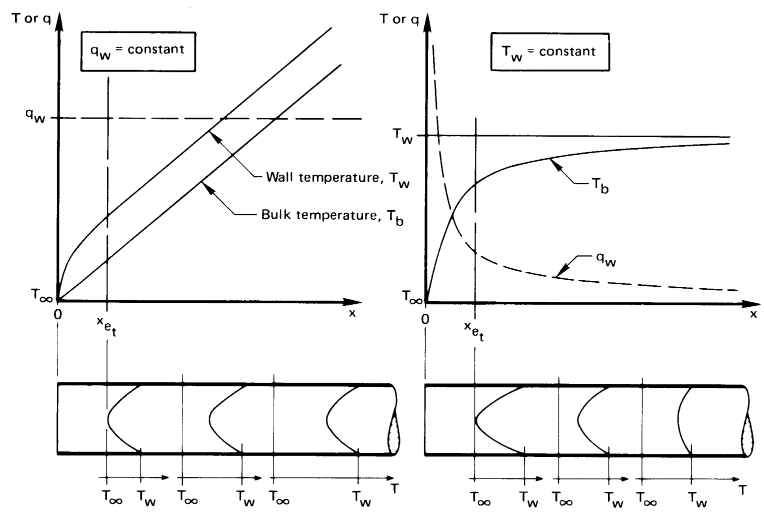

Figure 22: Temperature profile for constant heat flux and constant wall temperature in fully developed region

The thermal entrance length, x_{e_t} is the length after which the wall temperature becomes linear in a constant q_w case. For constant q_w, we have

$$\frac{\mathrm{d}T_w}{\mathrm{d}x} = \frac{\mathrm{d}T_b}{\mathrm{d}x} \tag{143}$$

In the fully developed region, shown in figure 22, for constant heat flux, we have

$$\frac{\partial}{\partial x}\left(\frac{T_w - T}{T_w - T_b}\right) = 0 \implies \frac{\partial}{\partial x}\left(\frac{T_w - T}{q_w/h}\right) = 0 \tag{144}$$

To get the $T(x,r)$ in a fully developed region with constant heat flux, we start with the energy equation in *cylindrical* coordinates:

$$\rho c_p \left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} \right) = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) \right)$$

in a fully developed region, there is $v = 0$. Additionally, $\frac{\partial^2 T}{\partial x^2} = 0$. Earlier, we defined

$$\frac{\partial T}{\partial x} = \frac{\mathrm{d}T_b}{\mathrm{d}x} = \frac{q_w p}{\dot{m} c_p}.$$

Using this, and the velocity profile equation 141, the energy equation now becomes

$$2\rho c_p u_{\text{avg}} \left[1 - \left(\frac{r}{R} \right)^2 \right] \frac{q_w 2\pi R}{\dot{m} c_p} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right).$$

We can integrate both sides to get a solution for temperature.

17 Lecture 17

Homework 8 is due this Friday. Our final exam will be during the final exam period: May 6th 9–11 AM.

From last class, the bulk temperature is given as T_b and is defined as 134, and can also be written as

$$T_b = \frac{\int_0^R \rho u c_p T 2\pi r \,\mathrm{d}r}{\dot{m} c_p}$$

for a cylindrical pipe. Note that c_p will cancel if it is constant. Also, ρ can also cancel with the $\dot{m}$, if it is constant.

We wil consider hydrodynamically and thermally fully developed flow with a constant wall heat flux. The analysis is based on the energy equation without time derivatives (since this is steady). We will still have convective derivative of the temperature:

$$\rho c_p \left(u\frac{\partial T}{\partial x} + \cancel{v\frac{\partial T}{\partial r}} \right) = k\left(\cancel{\frac{\partial^2 T}{\partial x^2}} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right) \right).$$

For constant wall flux, the temperature gradient w.r.t. x is simply 135; this is constant in laminar flow. The u is defined by 141. Substituting these two in, we get

$$2\rho \cancel{c_p} u_{\text{avg}} \left[1 - \left(\frac{r}{R} \right)^2 \right] \frac{q_w 2\pi R}{\dot{m} \cancel{c_p}} = k\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right).$$

Substituting $\dot{m} = u_{avg}\pi R^2$, we get

$$\frac{4\cancel{\rho u_{avg}\pi R} q_w}{k \cancel{\rho u_{avg}\pi R} R}\left(1 - \frac{r^2}{R^2} \right) = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r} \right)$$

Taking the integral w.r.t. r on both sides, we get

$$T(x,r) = \frac{4q_w}{Rk}\left(\frac{r^2}{4} - \frac{r^4}{16R^2} \right) + \cancel{C_1(x)}\ln r + C_2(x) \tag{145}$$

Note that the constants maybe dependent on x since this is the integral of a partial derivative. Since $r = 0 \implies T \to \infty$, $C_1(x)$ must be 0. Additionally, $C_2(x)$ must be dependent on the bulk temperature $T_b(x)$ which is a function of x.

$$T_b = \frac{\int_0^R uT2r \,\mathrm{d}r}{u_{avg} R^2}$$

where we will insert equation 141 for u, and 145 for T. Evaluating this integral will yield a 6th order polynomial, which is solvable since it is still a polynomial. We wwill eventually find

$$C_2(x) = T_b(x) - \frac{7}{24}\frac{q_w R}{k}.$$

Inserting this back into 145, we get

$$T(x,r) = T_b(x) + \frac{q_w R}{k}\left(\frac{r^2}{R^2} - \frac{1}{4}\frac{r^4}{R^4} - \frac{7}{24} \right). \tag{146}$$

This is shown in figure 23. At the wall, we get

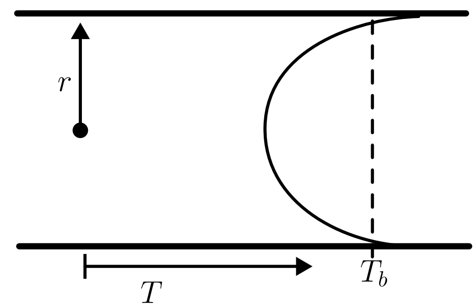

Figure 23: Temperature distribution for constant heat flux

$$T_w(x) = T_b(x) + \frac{q_w R}{k} \cdot \frac{11}{24}$$

and computing the heat flux, we have

$$q_w = \frac{24}{11} \cdot \frac{k}{R}(T_w - T_b)$$

which then yields

$$h = \frac{24}{11}\frac{k}{R} \tag{147}$$

which is independent of x. This ields a Nusselt number of

$$\mathrm{Nu}_D = \frac{48}{11}. \tag{148}$$

The book additionally shows the case where T_w is constant, which yields

$$\mathrm{Nu}_D = 3.657. \tag{149}$$

Exercise 7.1 shows a small pipe since a a small diameter is required for laminar flow. We should always check the Reynolds number first to ensure it is laminar. The problem is shown in figure 24. The calculated Reynolds number is 360 for water. We are asked to find the x at which the water temperature is 74°C at the hottest point (i.e., the wall temperature $T_w = 74$°C). When looking for the parameters of the fluid, ν in this case, we should

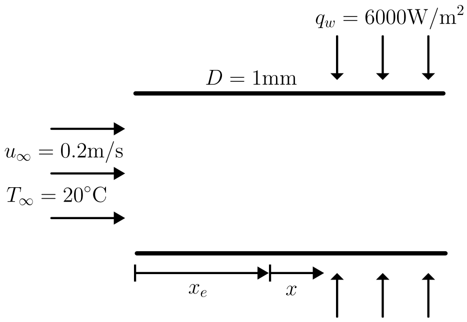

Figure 24: Exercise 7.1

use the film temperature, which is just the average temperature. In this case, this is

$$T_f = \frac{1}{2}(20° + 74°).$$

The bulk temperature, given by

$$T_b = \frac{q_w 2\pi R}{\dot{m} c_p} x + T_\infty$$

and the convection coefficient given by

$$h = \frac{48}{11} \cdot \frac{k}{D}$$

can be plugged into the equation

$$q_w = h(T_w - T_b)$$

to solve for T_w, which yields $x = 1.785\,\mathrm{m}$ after some re-arranging. Notice that the mass flow rate is very small. When the diameter is small, the h becomes very large. in this problem, the $h \approx 2800\,\mathrm{W/(m^2 \cdot K)}$. These micro-channels with high conductivity fluids such as liquid sodium can yield even higher heat conductivity. This has applications in micro-channel heat exchanger, such as for microelectronics. Micro heat transfer is a big field now. Additionally, this has laminar flow, which is easy to analyze.

The thermal entrance length x_{e_t} and the flow's entrance length x_e can differ, and the equation is

$$\frac{x_{e_t}}{D} = \begin{cases} 0.034\mathrm{Re}_D\mathrm{Pr} & T_w = \text{constant} \\ 0.043\mathrm{Re}_D\mathrm{Pr} & q_w = \text{constant} \end{cases} \tag{150}$$

for laminar flow only.

The heat transfer in the thermal entrance region is known as the Graetz problem. In this problem, we measure x from the beginning. The Nusselt number when it is developing and wall temperature is constant is given by

$$\mathrm{Nu}_D = 3.657 + \frac{0.0018\mathrm{Gz}^{1/3}}{\left(0.04 + \mathrm{Gz}^{-2/3}\right)^2} \tag{151}$$

in the book as equation 7.28, where

$$\mathrm{Gz} \equiv \mathrm{Re}_D\mathrm{Pr}\frac{D}{x} \tag{152}$$

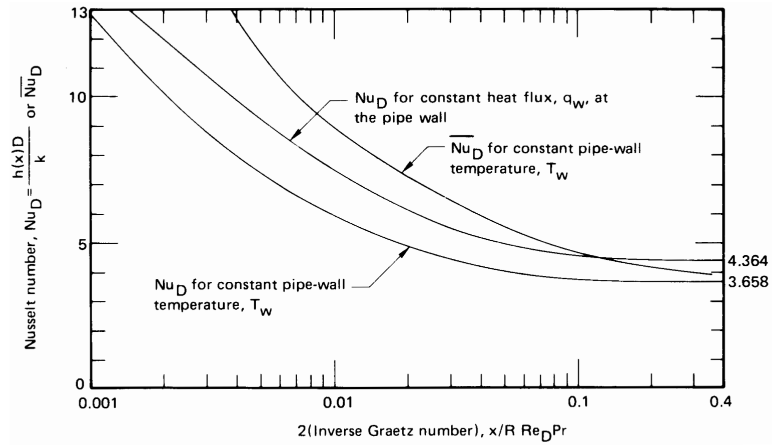

Figure 25: Graetz number, Figure 7.4 in book

and an example is shown in 25. This is as far as we can go in terms of analysis, since most fluid in pipe is turbulent, and consequently, numerical calculation and values are needed. However, internal pipe flow is simple to perform.

The flow becomes turbulent when the Reynolds number is about 2000. It is sometimes possible to still be laminar beyond this number, but this is only with *very smooth* pipes, which is most often not the case.

For turbulent flow, the Nusselt number is

$$\mathrm{Nu}_D = \frac{\mathrm{Re}_D\mathrm{Pr}(f/8)}{1 + 12.7\left(\mathrm{Pr}^{2/3} - 1\right)\sqrt{f/8}} \approx \mathrm{Re}_D\mathrm{Pr}^{1/3} \cdot \frac{f}{8} \tag{153}$$

where f is the *Darcy friction factor*, and we need to obtain this from the Moody chart. To use the Moody chart, we need the relative roughness and the Reynolds number. **Table 7.3** in book shows a wall roughness of some pipes, and it gets rougher over time due to fouling. There is also a 'modern formulation' of f (which is actually from the 1950s and 1960s), which gives

$$f = \frac{1}{(1.8\log_{10}\mathrm{Re}_D - 1.64)^2} \tag{154}$$

which is to be used with the equation

$$\mathrm{Nu}_D = \frac{\mathrm{Pr}\left(\mathrm{Re}_D - 1000\right)(f/8)}{1 + 12.7\left(\mathrm{Pr}^{2/3} - 1\right)\sqrt{f/8}} \tag{155}$$

and is applicable for Reynolds number of $2300 \le \mathrm{Re}_D \le 5 \times 10^6$ and $0.6 \le \mathrm{Pr} \le 1 \times 10^5$. Note that these are for *circular* pipes. Additionally, in our homework and other problems, we should first look at the Reynolds number before using these equations.

For non-circular pipes, we should utilize the hydraulic diameter and **Table 7.4** of the book. The mass flow rate calculation should be based on the actual pipe shape.

The cross-flow problem is the external problem. This is discussed in section 7.6 in the book. The Reynolds number determines the flow.

In chapter 8, we will consider natural convection, and condensation.

18 Lecture 18

Natural convection occurs during difference in temperature. Density of the fluid changes, which causes the fluid to rise. A boundary layer forms near the surface of the wall. We are interested in the temperature distribution in the boundary layer. This problem is shown in figure 26.

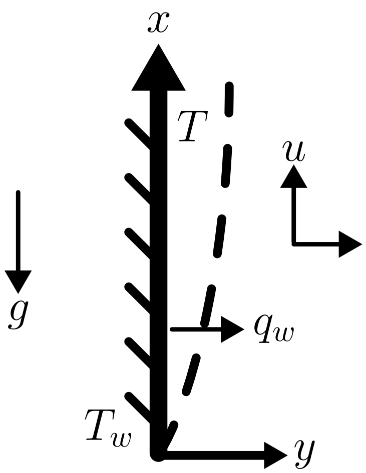

Figure 26: Convection problem

The x-momentum in the boundary layer now contains a gravity term:

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}-\rho g+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}\right)$$

In an open fluid, the pressure gradient inside the boundary layer is equivalent to the pressure gradient outside the boundary layer; this stems from the momentum equation in the y direction. This is equal to

$$\frac{\partial p}{\partial x}=-\rho_\infty g$$

which now leads to

$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=\rho_\infty g-\rho g+\mu\frac{\partial^2 u}{\partial y^2}$$

Transform this into

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=\underbrace{\left(\frac{\rho_\infty}{\rho}-1\right)g}_{\text{buoyancy}}+\nu\frac{\partial^2 u}{\partial y^2}$$

We relate the density term and temperature with the coefficient of thermal expansion, given as β in the book. The density near the wall is lower than further out. The definition for coefficient of thermal expansion is given as

$$\beta=-\frac{1}{\rho}\left(\frac{\partial\rho}{\partial T}_p\right) \tag{156}$$

Note that the subscript $\square_p$ denotes 'at constant pressure'. The coefficient is approximated as

$$\beta\approx\frac{1}{\rho}\frac{\rho-\rho_\infty}{T-T_\infty}$$

This is a linear approximation. We can rearrange this to be

$$\frac{\rho_\infty}{\rho}-1=\beta(T-T_\infty) \tag{157}$$

Plugging this into the x-momentum equation, we get

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=g\beta(T-T_\infty)+\nu\frac{\partial^2 u}{\partial y^2}$$

Unlike prior problems, this problem *explicitly couples* the temperature into the momentum equation.

The energy equation is written like prior problems:

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\left(\cancel{\frac{\partial^2 T}{\partial x^2}}+\frac{\partial^2 T}{\partial y^2}\right)$$

Typically, the third equation would be the continuity equation. There is a problem: here we can not assume incompressibility since density changes. However, we still use the incompressible assumption nonetheless:

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0.$$

If we wanted higher accuracy and had the option to do everything computationally, then we could use the compressible continuity equation. The book sticks to incompressible.

Using these three equations—x-momentum, energy, and continuity—we can find a similarity solution.

The integral equation for the x-momentum is slightly different due to the pressure term:

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u^2\,\mathrm{d}y=g\beta\int_0^\delta(T-T_\infty)\,\mathrm{d}y-\nu\left(\frac{\partial u}{\partial y}\right)_{y=0} \tag{158}$$

The heat flux equation by Fourier's law is

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta u(T-T_\infty)\,\mathrm{d}y=\frac{q_w}{\rho c_p}=-\alpha\left(\frac{\partial T}{\partial y}\right)_{y=0} \tag{159}$$

Note that the continuity equation is embedded in the above two.

The velocity u has an issue with scaling. In forced convection, there is a u_∞. In contrast, natural convection does not have one. We artificially create a characteristic scale, u_c. We have to find it in the problem. This transforms equation 158 to

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta\left(\frac{u}{u_c}\right)^2u_c^2\,\mathrm{d}y=g\beta\int_0^\infty\frac{T-T_\infty}{T_w-T_\infty}(T_w-T_\infty)\,\mathrm{d}y\ldots$$
$$-\nu u_c\left(\frac{\partial(u/u_c)}{\partial y}\right)_{y=0}$$

We re-arrange this into

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta\left(\frac{u}{u_c}\right)u_c\frac{T-T_\infty}{T_w-T_\infty}\cancel{(T_w-T_\infty)}\,\mathrm{d}y=$$
$$-\alpha\cancel{(T_w-T_\infty)}\left(\frac{\partial\frac{T-T_\infty}{T_w-T_\infty}}{\partial y}\right)_{y=0}$$

We assume that the non-dimensional temperature goes to zero *smoothly* at $y = \delta$. The temperature at $y = 0$ is T_w. Therefore, we write the temperature profile as

$$\frac{T - T_\infty}{T_w - T_\infty} = \left(1 - \frac{y}{\delta}\right)^2 = 1 - 2\left(\frac{y}{\delta}\right) + \left(\frac{y}{\delta}\right)^2 \tag{160}$$

Note that because we are using the integral method, we can use such an incorrect guess for the temperature profile.

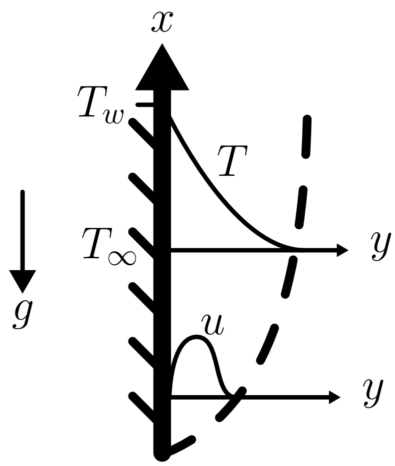

Figure 27: Temperature and velocity profile in natural convection

The velocity profile is

$$\frac{u}{u_c} = \frac{y}{\delta}\left(1 - \frac{y}{\delta}\right)^2 \tag{161}$$

and both are shown in figure 27 and plugging this into our integral equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{u_c^2 \delta \int_0^1 \left(\frac{u}{u_c}\right)^2 \mathrm{d}\left(\frac{y}{\delta}\right)\right\}$$
$$= g\beta(T_w - T_\infty)\delta \int_0^1 \left(\frac{T - T_\infty}{T_w - T_\infty}\right) \mathrm{d}\left(\frac{y}{\delta}\right) - \frac{\nu u_c}{\delta}$$

This simplifies the heat flux equation to

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{u_c \delta \int_0^1 \left(\frac{u}{u_c}\right) \frac{T - T_\infty}{T_w - T_\infty} \mathrm{d}\left(\frac{y}{\delta}\right)\right\} = -\frac{\alpha}{\delta} \cdot -2$$

and computing this would yield a number for the x-momentum integral equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{u_c^2 \delta}{105}\right\} = g\beta(T_w - T_\infty) \cdot \frac{\delta}{3} - \frac{\nu u_c}{\delta}.$$

and for the energy equation:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{u_c \delta}{30}\right\} = \frac{2\alpha}{\delta}$$

The above two are 2 coupled non-linear ordinary differetial equations for $\delta(x)$ and $u_c(x)$. We *cannot assume* that u_c is constant; this will yield an inconsistency between the two equations.

The solution was found by Tryal (this is the old English word for 'trial'). We set

$$u_c = C_1 x^m$$
$$\delta = C_2 x^n$$

Plugging these into the two equations, we get

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{C_1^2 C_2 x^{2m} x^n}{105}\right\} = g\beta(T_w - T_\infty)\frac{C_2 x^n}{3} - \nu \frac{C_1}{C_2} x^{m-n}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{\frac{C_1 C_2 x^{m+n}}{30}\right\} = 2\alpha \frac{x^{-n}}{C_2}.$$

Taking the derivatives w.r.t. x, we get

$$(2m+n)\frac{C_1^2 C_2 x^{2m+n-1}}{105} = g\beta(T_w - T_\infty)\frac{C_2 x^n}{3} - \nu \frac{C_1}{C_2} x^{m-n}$$
$$(m+n)\frac{C_1 C_2 x^{m+n-1}}{30} = 2\alpha \frac{x^{-n}}{C_2}.$$

In the above, the only way that the left and right hand side can be equal is if the following are met:

$$2m + n - 1 = n = m - n$$
$$m + n - 1 = -n$$

This is an interesting problem since there are 3 equations for 2 unknowns. However, this does have a solution:

$$m = \frac{1}{2}$$
$$n = \frac{1}{4}.$$

This makes sense: in studies of natural convection, the boundary layer grows to the $x^{1/4}$. Knowing the powers, we need to determine the constants. The process is

$$\frac{5}{4 \cdot 105} C_1^2 C_2 = g\beta(T_w - T_\infty)\frac{C_2}{3} - \nu \frac{C_1}{C_2}$$
$$\frac{3}{4 \cdot 30} C_1 C_2 = \frac{2\alpha}{C_2}$$

which yields

$$C_1 = 5.17\nu \left(\frac{20}{21} + \frac{\nu}{\alpha}\right)^{-1/2} \left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{1/2} \tag{162}$$

$$C_2 = 3.93 \left(\frac{20}{21} + \frac{\nu}{\alpha}\right)^{1/4} \left(\frac{g\beta(T_w - T_\infty)}{\nu^2}\right)^{-1/4} \left(\frac{\nu}{\alpha}\right)^{-1/2}. \tag{163}$$

Notice that the Prandtl number appears here. Plugging this to solve for heat flux at the wall, we get

$$q_w = 2k \frac{T_w - T_\infty}{\delta} \tag{164}$$

where we know δ now using the $C_2 x^n$. Notice that the heat flux in the body and convection must be equal, so

$$h(T_w - T_\infty) = \frac{2k}{\delta}(T_w - T_\infty)$$

which yields

$$h = \frac{2k}{\delta} \tag{165}$$

and the Nusselt number is simply

$$\mathrm{Nu}_x = \frac{2x}{\delta}. \tag{166}$$

Since

$$\frac{\delta}{x} = C_2 x^{-3/4}$$

we can re-write this as

$$\frac{\delta}{x} = 3.93\left(\frac{20}{21} + \mathrm{Pr}\right)^{1/4}\mathrm{Pr}^{-1/2}\left(\frac{g\beta(T_w - T_\infty)x^3}{\nu^3}\right)^{-1/4}$$

We did this to get a non-dimensional number in all of the terms. The last term above is the Grashof number:

$$\boxed{\mathrm{Gr}_x = \frac{g\beta(T_w - T_\infty)x^3}{\nu^2}}. \tag{167}$$

This transforms the Nusselt number to

$$\mathrm{Nu}_x = \frac{2}{3.93}\left(\frac{20}{21} + \mathrm{Pr}\right)^{-1/4}\mathrm{Pr}^{1/2}\mathrm{Gr}_x^{1/4} \tag{168}$$

which is now dimensionless in all terms. More numerically and compact is

$$\mathrm{Nu}_x = 0.508\left(\frac{0.952 + \mathrm{Pr}}{\mathrm{Pr}}\right)^{-1/4}\left(\underbrace{\mathrm{PrGr}_x}_{\mathrm{Ra}_x}\right)^{1/4} \tag{169}$$

and is **Eq 8.25** in the book.

The comparison between the integral method and the numeric calculation is shown in 28, which is **Fig 8.4** in the book.

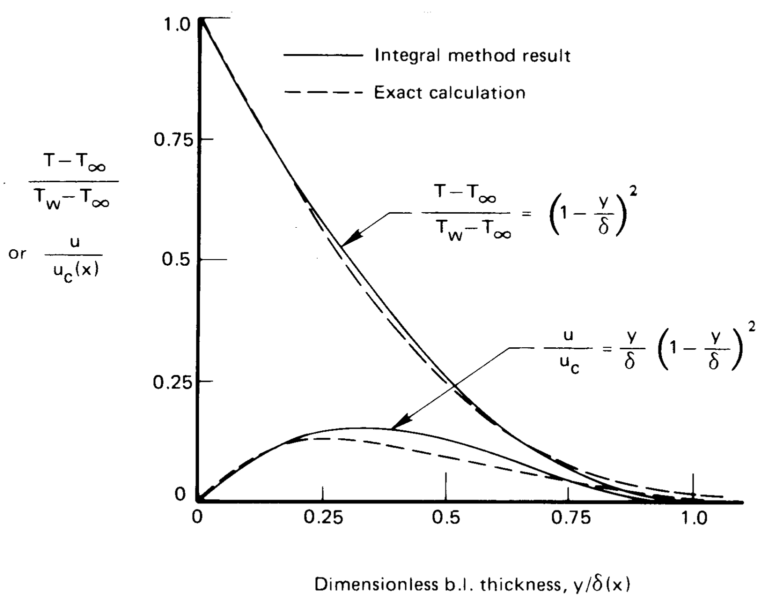

Figure 28: Comparing integral solution with numeric solution

The average convection coefficient is given by

$$\bar{h} = \frac{1}{L}\int_0^x h(x)\,\mathrm{d}x$$

which then translates to

$$\overline{\mathrm{Nu}}_L = \frac{\bar{h}L}{k}.$$

The 'average' is more practical since most practical concerns are for the net heat transfer. The average Nusselt number was experimentally found to be

$$\boxed{\overline{\mathrm{Nu}}_L = 0.68 + 0.67\mathrm{Ra}_L^{1/4}\left[1 + \left(\frac{0.492}{\mathrm{Pr}}\right)^{9/16}\right]^{-4/9}} \tag{170}$$

where the Rayleigh number is

$$\mathrm{Ra}_x = \mathrm{PrGr}_x. \tag{171}$$

19 Lecture 19

Homework 9 is due Friday.

In the natural convection problem, we typically assume the surface to be hot, and the air to be cold. In the natural convection problem, we assume that

$$\delta(x) \approxeq \delta_t(x)$$

and this is sometimes not true, especially when the fluid is very viscous which causes the cold fluid to be 'dragged along'.

When we evaluate the fluid properties such as ν, we use the film (i.e., average between wall and fluid) temperature. However, for β, use the *fluid* temperature!

For figure 27, we used the x-momentum and energy equations to derive equation 169. Using the Nusselt number, we see that the dependency is

$$h(x) \propto x^{-1/4}(T_w - T_\infty)^{1/4}$$

which indicates that, at the leading edge, there is a singularity. Similarly, the heat flux is

$$q_w(x) \propto x^{-1/4}(T_w - T_\infty)^{5/4}$$

which is important because this showcases that the dependency is not linear. In natural convection, $h(x)$ itself depends on the temperature difference. If we wanted the net heat transfer, we have to integrate over the vertical distance.

We derived equation 170 by taking the average of the convection coefficient:

$$\bar{h} \propto \frac{1}{L}\int_0^L x^{-1/4}\,\mathrm{d}x \propto \frac{4}{3}L^{-1/4}.$$

So far, these are constant wall temperature problems. If we had an electric heater on the plate, it would be more of a constant heat flux problem. Shown in figure 29, we get the Nusselt number to instead be

$$\mathrm{Nu}_x = 0.630\left(\frac{\mathrm{Ra}_x^*\mathrm{Pr}}{4 + 9\sqrt{\mathrm{Pr}} + 10\mathrm{Pr}}\right)^{1/5} \tag{172}$$

where

$$\mathrm{Ra}_x^* = \frac{g\beta q_w x^4}{k\nu\alpha} \tag{173}$$

and the average Nusselt number is similarly found to be

$$\overline{\mathrm{Nu}}_L = \frac{6}{5}\cdot 0.630\left(\frac{\mathrm{Ra}_x^*\mathrm{Pr}}{4+9\sqrt{\mathrm{Pr}}+10\mathrm{Pr}}\right)^{1/5}. \tag{174}$$

Note that, this is **Eq 8.44b** in the book, but that has a typo.

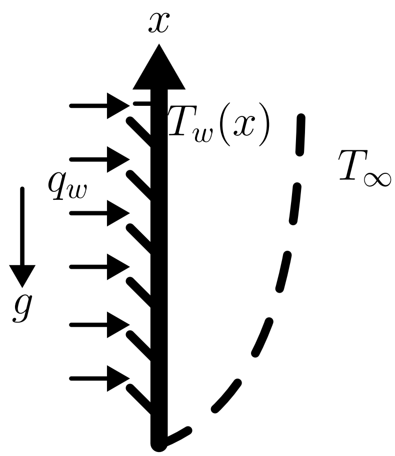

Figure 29: Constant heat flux convection

The Grashof is the analogy of the Reynolds number. The buoyancy drives the fluid flow, and is the driving force of convection. There is no more u_∞ in the Grashof number, and it determines the transition between laminar and turbulent flow.

The empirical turbulent Nusselt number for constant wall temperature *and* constant heat flux is given in **Eq 8.13b**:

$$\overline{\mathrm{Nu}}_L = \left\{0.825 + \frac{0.387\,\mathrm{Ra}_L^{1/6}}{\left[1+(0.492/\,\mathrm{Pr})^{9/16}\right]^{8/27}}\right\}^2 \tag{175}$$

When the flat plate is a cylinder instead, there is a correction factor shown in the book's **Fig 8.7**. For these, there is also a difference between laminar and turbulence flow, shown in equations **Eq 8.28** and **Eq 8.29**. Note that, in cylindrical and spherical problems, the Nusselt number has a dependency on *diameter*.

Turbulence onsets at $\mathrm{Gr}_L \geq 10^9$.

Note that, for spherical and cylindrical problems, there is a constant term, which allows for heat transfer even in the absence of gravity.

When the plate is tilted, we include tho component of gravity $g\cos(\theta)$; this works in most cases, though there is a big difference between the bottom of the plate (which looks a lot like figure 27), but the top surface has updrafts/plumes; for cold plate in hot fluid, it is opposite. These happens because the fluid is free to get away, and becomes unstable, and separates.

In a completely horizontal plate, the existing Rayleigh number is no longer valid; the flow on the bottom behaves more like a sphere, and the flow on the top completely acts like plumes. This problem looks a lot more like natural convection on earth's surface. The opposite is true for a cold plate. The tendency for fluid to form plumes is a fluid mechanics problem.

In film condensation, there is a saturated vapor, that is, something that is almost ready to turn from gas to liquid stage. In this problem, the plate is typically cold, and fluid is hot. The liquid, instead, forms the boundary layer. This is shown in figure 30. Here,

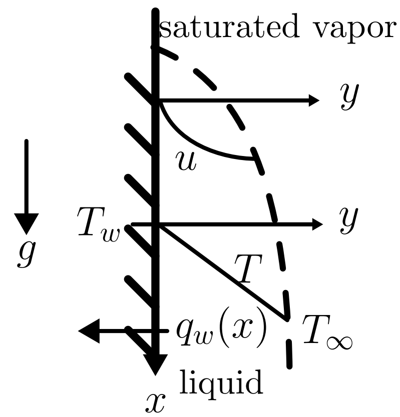

Figure 30: Film condensation

our x-momentum equation is now

$$\rho_f\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x}+\rho_f g+\mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}+\frac{\partial^2 u}{\partial y^2}\right)$$

where

$$\frac{\partial p}{\partial x} = \rho_g g$$

which lets us write

$$\begin{aligned}\rho_f\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right) &= -\rho_g g+\rho_f g+\mu\frac{\partial^2 u}{\partial y^2}\\ u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} &= g\left(\frac{\rho_f-\rho_g}{\rho_f}\right)+\nu\frac{\partial^2 u}{\partial y^2}.\end{aligned}$$

The energy equation only has heat transfer in the y-direction as

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}.$$

This can be solved via differential method, integral method, or Tryal.

Nusselt himself solved this by ignoring convective terms; this is applicable since this is primarily the fluid creeping down, and not fluid rising up; the flow is very slow, so this is possible. By ignoring this, we ignore the non-linear terms, which makes it very easy to solve the problem. This turns the x-momentum equation to be gravity dependent only:

$$\frac{\partial^2 u}{\partial y^2} = -\frac{g}{\nu}\frac{\rho_f-\rho_g}{\rho_f}$$

which can be integrated twice, and knowing that $\mu = \nu\rho_f$:

$$u = -\frac{g}{\mu}\frac{(\rho_f-\rho_y)}{2}y^2+C_1 y+C_2.$$

Since $u = 0$ at $y = 0$, we have $C_2 = 0$. Next, at the boundary layer $y = \delta$, the gradient is $\frac{\partial u}{\partial y} = 0$, which leads to

$$C_1 = \frac{g}{\mu}(\rho_f - \rho_g)\delta.$$

After a couple of additional steps, we get

$$u = \frac{(\rho_f - \rho_g)g\delta^2}{2\mu}\left[-\left(\frac{y}{\delta}\right)^2 + 2\left(\frac{y}{\delta}\right)\right] \tag{176}$$

and the energy equation can next be transformed into

$$0 = \cancel{\alpha}\frac{\partial^2 T}{\partial y^2}$$
$$T = C_1 y + C_2$$

The boundary conditions are T_w at the wall and $T_\infty = T_{\text{sat}}$ at the boundary layer, which leads to

$$T(y) = T_w + (T_{\text{sat}} - T_w)\frac{y}{\delta} \tag{177}$$

where we are left to find δ. We obtain δ via the energy balance and continuity.

When vapor condenses, it releases energy. The mass and energy balance are shown in figures 31 and 32 respectively. The mass

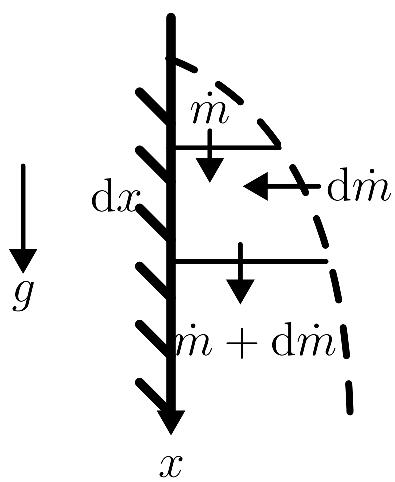

Figure 31: Film condensation — mass balance

flow rate is equivalent to

$$\dot{m} = \int_0^\delta \rho_f u \, \mathrm{d}y$$

which can be made to have the similarity concept:

$$\dot{m} = \int_0^1 \rho_f u \, \mathrm{d}\left(\frac{y}{\delta}\right)\delta.$$

We can substitute equation 176 into the above equation to find the mass flow rate. Performing this, we get

$$\dot{m} = \frac{\rho_f(\rho_f - \rho_g)g\delta^3}{3\mu}. \tag{178}$$

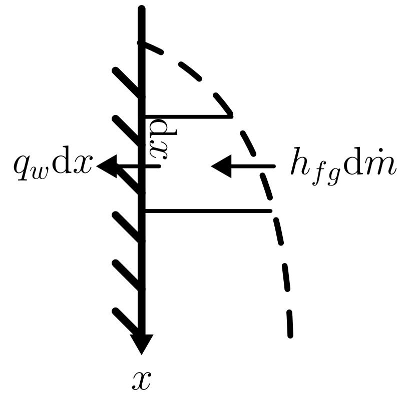

Figure 32: Film condensation — energy balance

To find d$\dot{m}$, we simply take the derivative w.r.t. δ:

$$\mathrm{d}\dot{m} = \frac{\rho_f(\rho_f - \rho_g)g}{3\mu}3\delta^2 \, \mathrm{d}\delta.$$

We plug this into the temperature differential equation

$$q_w \, \mathrm{d}x = h_{fg} \, \mathrm{d}\dot{m}.$$

This equation is coming from energy balance equation shown in figure 32: Here, $q_w = -k\left(\frac{\partial T}{\partial y}\right)_{y=0}$, where

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = \frac{T_{\text{sat}} - T_w}{\delta}.$$

Plugging the d$\dot{m}$ equation into this, we get

$$-k\left(\frac{T_{\text{sat}} - T_w}{\delta}\right)\mathrm{d}x = \frac{h_{fg}\rho_f(\rho_f - \rho_g)g}{\mu}\delta^2 \, \mathrm{d}\delta.$$

Note that, as shown in figure 30, we *must not use a negative sign*, because the direction of q_w is already assumed to be in the opposite direction. Also, note that ρ_f is density of the liquid.

In the above equation, moving dx to the right hand side, and then cancelling δ once, we get

$$\frac{k\mu(T_{\text{sat}} - T_w)}{h_{fg}\rho_f(\rho_f - \rho_g)g} = \frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x}\delta^4.$$

Integrating w.r.t. x, we get

$$\boxed{\delta(x) = \left[\frac{4ku(T_{\text{sat}} - T_w)}{h_{fg}\rho_f(\rho_f - \rho_g)g}\right]^{1/4} x^{1/4}} \tag{179}$$

where the constant of integration $C = 0$ since the boundary layer thickness is 0 at $x = 0$. Using this, we find an equation for the heat flux:

$$q_w = +k\left(\frac{\partial T}{\partial y}\right)_{y=0} = k\frac{\cancel{T_{\text{sat}} - T_w}}{\delta} = h\cancel{(T_{\text{sat}} - T_w)}$$

which gives us h as well. This enables us to find the Nusselt number:

$$\boxed{\mathrm{Nu}_x = \left[\frac{h_{fg}\rho_f(\rho_f - \rho_g)gx^3}{4k\mu(T_{\text{sat}} - T_w)}\right]^{1/4}} = \frac{hx}{k}. \tag{180}$$

20 Lecture 20

In the beginning of the condensation process, there's drops forming. Later we have a flow down the surface. This is because, with enough drops, they always want to coales. From the Nusselt analysis, last class for constant wall temperature we had

$$\mathrm{Nu}_x = 0.707\left[\frac{\rho_f(\rho_f-\rho_g)gh_{fg}x^3}{\mu k(T_{\mathrm{sat}}-T_w)}\right]^{1/4}$$

where all the properties are fluid except ρ_g and h_{fg}.

To get a better agreement with experiments, a corrected latent heat is used:

$$h'_{fg} = h_{fg}\left[1+\left(0.683-\frac{0.228}{\mathrm{Pr}}\right)\mathrm{Ja}\right] \tag{181}$$

where

$$\mathrm{Ja} \equiv \frac{c_p(T_{\mathrm{sat}}-T_w)}{h_{fg}} \tag{182}$$

is the Jakob number.

In example 8.6, we have water at $T_{\mathrm{sat}} = 100°C$. The wall is at a constant $T_w = 90°C$. A constant wall temperature is feasible such as by having constant cooling water flowing on the other side of the wall; this is done in power plants, which dumps the now-warmed water back to rivers. The height of the wall is given as $L = 0.3\,\mathrm{m}$. We are asked to find the heat transfer assuming that the wall is 1 meter thick. The total condensation region δ_L also needs to be found for a meter depth.

To solve this, we have to use equation 180 to find the heat transfer followed by finding Q, and 179 for the boundary layer thickness. The properties should be evaluated at the film temperature (average temperature), except for ρ_g and h_{fg} which should be evaluated at the saturation temperature. We can use both h_{fg} or calculate h'_{fg} and see the difference in accuracy. In this problem, the accuracy is already fairly good.

For the Q, we should use the average h, by averaging $h(x)$ which is obtained from the Nusselt number. Even though the temperature difference is not large, there is a large heat transfer of $26\,\mathrm{kW}$ for unforced convection, because the energy due to change of phase is large.

Another way to find the heat flux is $q_w \mathrm{d}x = h_{fg}\mathrm{d}\dot{m}$. When we integrate this, we get the total $Q = h_{fg}\dot{m}$. This makes sense Q has the units of $\left[\frac{\mathrm{kJ}}{\mathrm{s}}\right]$, while $h_{fg}\dot{m}$ has $\left[\frac{\mathrm{kJ}}{\mathrm{kg}}\cdot\frac{\mathrm{kg}}{\mathrm{s}}\right]$. The mass transfer rate is $\dot{m} = 0.01\,\mathrm{kg/s}$, which is small. Real condensers, consequently, need a large surface area.

We can use a similarity solution in the liquid boundary layer. To obtain this, we assume incompressible continuity (this is realistic *within* the liquid):

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y} = 0$$

and the x-momentum equation

$$u\frac{\partial u}{\partial x}+v\frac{\partial v}{\partial y} = \frac{\rho_f-\rho_g}{\rho_f}g+\nu\frac{\partial^2 u}{\partial y^2}$$

and the energy equation

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2}$$

The similarity variable discovered is

$$\eta = \left(\frac{(\rho_f-\rho_g)g}{4\nu^2\rho_f}\right)^{1/4} yx^{-1/4} = Cyx^{-1/4}$$

Then, the stream function is set to

$$\psi = 4\nu Cx^{3/4}f(\eta)$$

which yields the velocity components to be

$$\begin{aligned} u &= \frac{\partial\psi}{\partial y} && = 4\nu C^2x^{1/2}f'(\eta) \\ v &= -\frac{\partial\psi}{\partial x} && = -\nu Cx^{-1/4}\left[3f(\eta)-\eta f'(\eta)\right]. \end{aligned}$$

This transformation naturally satisfies the continuity equation. The x-momentum equation becomes

$$f''' + 3ff'' - 2f^{'2} + 1 = 0$$

We solve this ordinary differetial equation using the boundary conditions:

- $u(y=0)=0 \implies f'(\eta=0)=0$ (no-slip condition, use on u)
- $v(y=0)=0 \implies f(\eta=0)=0$ (no-slip condition, use on v)
- $\mu\frac{\partial u}{\partial y}\Big|_{y=\delta} = 0$ is the no shear condition at the fluid-gas interface. This yields $f''(\eta = C\delta x^{-1/4} = \eta_\delta) = 0$.

A shooting method is required for this. However, we don't know η_δ; hence, we have to do the shooting method for multiple values of η_δ. We keep track of the results.

We then use the energy equation, where we plug in

$$\Theta(\eta) = \frac{T-T_{\mathrm{sat}}}{T_w-T_{\mathrm{sat}}}$$

and its derivatives. This converts the energy equation into

$$\Theta'' + 3\mathrm{Pr}f(\eta)\Theta' = 0.$$

For this, we have the boundary conditions

- $T(y=0)=T_w \implies \Theta(\eta=0)=1$
- $T(y=\delta)=T_{\mathrm{sat}} \implies \Theta(\eta=\eta_\delta)=0$

Here, we can plug in $f(\eta)$ from the x-momentum solution. We have to perform the shooting method again, for several η_δ.

The final step involves finding a relationship between the shooting problem solutions and η_δ. This is done by an energy equation at the interface. At the interface, energy liberated by condensation is equal to the heat flux at the interface:

$$q_{int}\,\mathrm{d}x = h_{fg}\,\mathrm{d}\dot{m}.$$

Notice that the left hand side will have a solution in terms of Θ due to temperature dependence, and the right hand side will have only f dependence due to continuity nature. This yields:

$$k\left(\frac{\partial T}{\partial y}\right)_{y=\delta} = h_{fg}\frac{\mathrm{d}\dot{m}}{\mathrm{d}x}$$

which can start non-dimensionalizing by

$$k(T_w - T_{\mathrm{sat}})\Theta'(\eta_\delta)Cx^{-1/4} = h_{fg}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\delta \rho u\,\mathrm{d}y$$

where the $\dot{m}$ component simplifies to

$$\int_0^{\eta_\delta} \rho(4\nu C^2 x^{1/2} f'(\eta))\frac{x^{1/4}}{C}\,\mathrm{d}\eta \Rightarrow \dot{m} = \rho 4\nu C x^{3/4} f(\eta_\delta).$$

Inserting this back, we get

$$k(T_w - T_{\mathrm{sat}})\Theta'(\eta_\delta)Cx^{-1/4} = h_{fg}\cdot 3\rho\nu C x^{-1/4} f(\eta_\delta)$$

A lot of the term cancels, and we get

$$k(T_w - T_{\mathrm{sat}})\Theta'(\eta_\delta) = h_{fg}\cdot 3\rho\nu f(\eta_\delta)$$

where Θ' and f are known from the shooting method, at η_δ.

When we are given a vertical cylinder, we can still assume flat plate if $\delta \ll R$. However, if the radius of the cylinder is small (such as very thin wires), a different solution is needed. When the cylinder is horizontal, the average Nusselt number has to be used. Average also has to be used for a sphere.

The fluid flow in a condensation can also be turbulent. Instability happens when the condensation Reynolds number reaches 7:

$$\mathrm{Re}_c = \frac{\dot{m}}{\mu} = \frac{\rho u_{\mathrm{avg}}\delta}{\mu}. \tag{183}$$

Transition to turbulence happens at $\mathrm{Re}_c \approx 400$.

Electromagnetic radiation happens so long as temperature is above 0 K. The radiation does not require a medium. The net heat exchange due to radiation depends on

- Temperatures of two bodies
- Surface area
- Geometry and orientation and spacing
- Spacing of the surface
- Reflectinos off of other surfaces
- Presence of a medium.

As technology developed, we became more and more capable in producing shorter wavelength radiations. In this case, thermal radiation is considered to be visible, infrared, and UV.

The peak of the solar radiation happens at the visible range.

21 Lecture 21

In the book, $e_\lambda(\lambda, T)$ is the monochromatic emissive power. In other words, it is

$$e_\lambda(\lambda, T) = \frac{\text{energy}}{\text{time} \times \text{area} \times \text{wavelength}} \tag{184}$$

but notice that it is not really a 'power'. To find the energy in a band, you integrate between the wavelengths and is known as a power density. This can yield the total emissive power

$$e(T) = \int_0^\infty e_\lambda(\lambda, T)\,\mathrm{d}\lambda. \tag{185}$$

A black body is a body that absorbs all incident radiation. There are no real black bodies in nature, but approximate ones do exist. At thermal equilibrium, a black body must also emit radiation. We can measure the emitted radiation.

A spherical hohlraum in a sphere with a small apperture; the sphere is at a constant temperature. Any radiation that goes through the hole has a much harder time getting out: the hole is a nearly perfect blackbody. Wien theorized a model that did not work. Planck quantized the radiation, and doing so enabled him to get a very good agreement with the data, better than Wien. Initially he did not publish this work since quantized energy seemed ridiculous. When he finally did publish, it led to the development of quantum mechanics. The Planck's constant comes from fitting the data of the monochromatic emmissive power.

The book uses the monochromatic black body radiation:

$$e_{\lambda b}(\lambda, T) = \frac{2\pi h c_o^2}{\lambda^5\left[\exp\left(\frac{hc_0}{k_B T\lambda}\right) - 1\right]} \tag{186}$$

The sun emmits a seemingly black body radiation, and measuring the spectrum allows estimating the temperature. Integrating along the wavelength yields the power density of a black body showcases the Stefan-Boltzmann constant

$$e_b(T) = \int_0^\infty e_{\lambda b}(\lambda, T)\,\mathrm{d}\lambda = \sigma T^4 \ \ni \sigma = 5.67\times 10^{-8}\,\frac{\mathrm{W}}{\mathrm{m^2K}}. \tag{187}$$

Although real bodies typically don't emit like black bodies, we can compare their radiation to them. For a nonblack body, we can define the monochromatic emittance, ε_λ:

$$\varepsilon_\lambda = \frac{\epsilon_\lambda(\lambda, T)}{e_{\lambda b}(\lambda, T)}. \tag{188}$$

Taking the integral yields the total emittance:

$$\varepsilon = \frac{\int_0^\infty e_\lambda(\lambda, T)\,\mathrm{d}\lambda}{\int_0^\infty e_{\lambda b}(\lambda, T)\,\mathrm{d}\lambda}. \tag{189}$$

Note that this is very much dependent on the surface of the body. Polished aluminum has very little radiation: it also has very little absorption. Red brick has very high emittance and absorptance. Burnt carbon has an emittance of 0.95. The material does not have to be black in color: rough concrete has 0.94. Ice interestingly has a high emittance, up to 0.98.

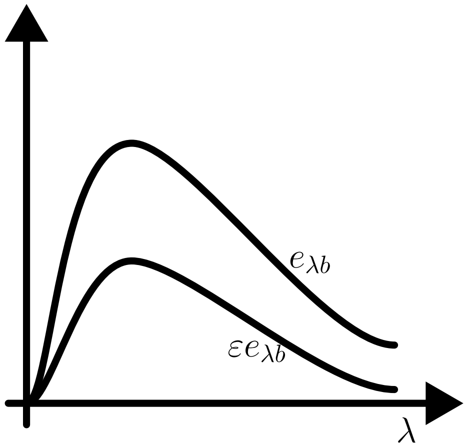

Figure 33: Gray body comparison

A gray body is a material where $\varepsilon_\lambda = \varepsilon$ for all wavelengths. Figure 33 shows this, and the equation for gray body is given as

$$e(T) = \varepsilon\sigma T^4.$$

The simplest radiation heat transfer problem is two parallel black planes. We assume there is no reflections. In such a case:

$$q_{net,1\to 2} = e_b(T_1) - e_b(T_2) = \sigma(T_1^4 - T_2^4)$$

and is shown in 34.

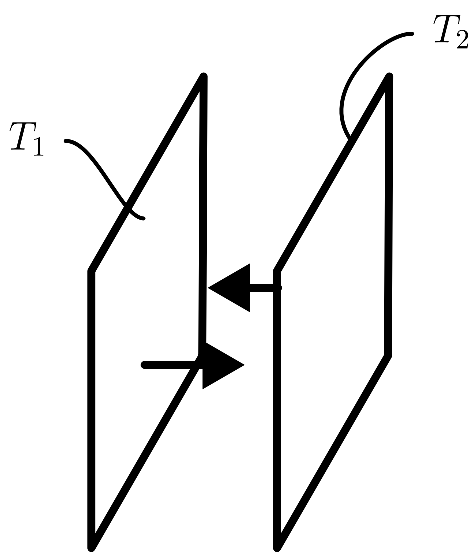

Figure 34: Parallel bodies

$\mathrm{d}\omega$ is the solid angle subtended by $\mathrm{d}A$ with units of steradians. The outgoing radiation by the small area is

$$\mathrm{d}Q_{out} = i_b \,\mathrm{d}\omega \,\mathrm{d}A \cos\theta = i_b \frac{\mathrm{d}A_a}{r^2}\,\mathrm{d}A\cos\theta$$

and is illustrated in figure 35. The heat flux is simply

$$\mathrm{d}q_{out} = i_b \frac{\mathrm{d}A_a}{r^2}\cos\theta.$$

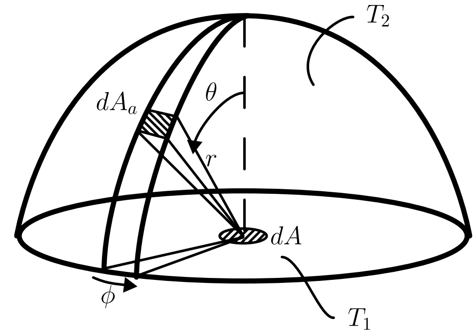

Figure 35: Energy intensity to a hemispherical shell

To get the true heat, we integrate over the hemisphere:

$$\begin{aligned} e_b &= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} i_b r^2 \sin\theta \,\mathrm{d}\theta\,\mathrm{d}\phi \\ &= i_b 2\pi \int_0^{\pi/2} \sin\theta\cos\theta\,\mathrm{d}\theta \\ &= i_b\pi \end{aligned}$$

where i_b is the intensity of the emmission, and yields the definition

$$i_b = \frac{e_b}{\pi}.$$

Here, the $\mathrm{d}\omega_1$ is the solid angle subtended by $\mathrm{d}A_2$ as seen from $\mathrm{d}A_1$ and is given by

$$\frac{\mathrm{d}A_2\cos\beta_2}{s^2} = \mathrm{d}\omega_1.$$

where s is the distance from one patch to the other. Using figure 36, we have the net heat

$$\begin{aligned} \mathrm{d}Q_{1\to 2} &= i_1\,\mathrm{d}\omega_1\,\mathrm{d}A_1\cos\beta_1 \\ &= i_1 \frac{\mathrm{d}A_2\cos\beta_2\,\mathrm{d}A_1\cos\beta_1}{s^2} \\ \mathrm{d}Q_{2\to 1} &= i_2 \frac{\mathrm{d}A_1\cos\beta_1\,\mathrm{d}A_2\cos\beta_2}{s^2} \end{aligned}$$

To get the net heat, we have to integrate. However, this time the angles β_1 and β_2, and the lengths s changes:

$$\begin{aligned} \mathrm{d}Q_{1\to 2,net} &= \mathrm{d}Q_{1\to 2} - \mathrm{d}Q_{2\to 1} \\ &= (i_1 - i_2)\frac{\cos\beta_1\cos\beta_2\,\mathrm{d}A_1\,\mathrm{d}A_2}{s^2} \\ Q_{1\to 2,net} &= \frac{\sigma(T_1^4 - T_2^4)}{\pi}\int_{A_2}\int_{A_1}\frac{\cos\beta_1\cos\beta_2\,\mathrm{d}A_1\,\mathrm{d}A_2}{s^2} \\ Q_{2\to 1,net} &= \frac{\sigma(T_2^4 - T_1^4)}{\pi}\int_{A_1}\int_{A_2}\frac{\cos\beta_1\cos\beta_2\,\mathrm{d}A_1\,\mathrm{d}A_2}{s^2} \end{aligned}$$

We simply this by considering the view factor, which absorbs the integral:

$$Q_{1\to 2,net} = A_1 F_{12}\sigma(T_1^4 - T_2^4) \tag{190}$$

where

$$F_{12} = \frac{1}{A_1}\int_{A_1}\int_{A_2}\frac{\cos\beta_1\cos\beta_2}{\pi s^2}\,\mathrm{d}A_1\,\mathrm{d}A_2. \tag{191}$$

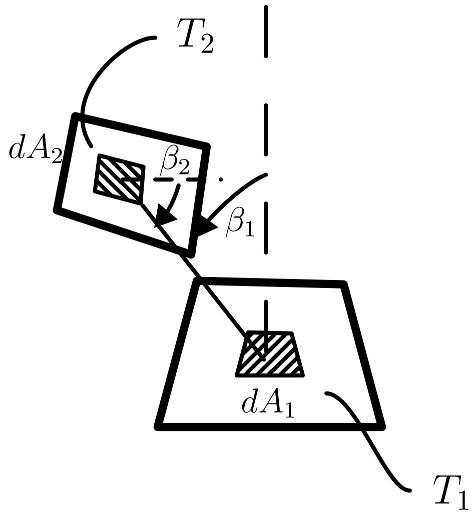

Figure 36: Energy intensity to an inclined plane

The heat from $2 \to 1$ will only swap out the subscripts. Note that, when the bodies are not black, we have to use a different technique.

Here, we notice a reciprocity property:

$$A_1 F_{12} = A_2 F_{21}$$

which is the fraction of a field of view of 1 that is occupied by 2.

Note that it is possible to have a view factor to one's own body, such as in a concave body, such as in figure 37.

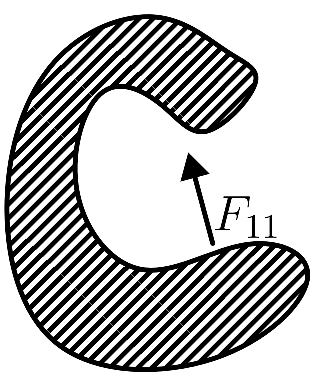

Figure 37: View factor of a concave body

As objects get closer, the view factors increase. **Table 10.2** discusses some of the view factors for 2 dimensional objects.

22 Lecture 22

A black body does not radiate only if the temperature is at absolute zero. The emissive intensity is given by

$$i_b = \frac{e_b}{\pi} \left[\frac{\mathrm{W}}{\mathrm{m}^2 \cdot \mathrm{steradian}}\right]. \tag{192}$$

This is used for the net heat transfer, which uses a view factor. The β in figures 36 is the angle between the normal of that plane and the straight line from the plane to the other plane. The view

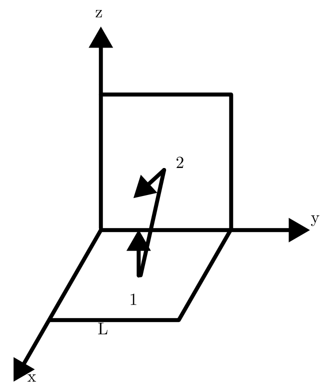

Figure 38: Example plane problem

factor F_{12} is non-dimensional. In Figure 38, the distance between the two plane differentials is

$$s^2 = x_1^2 + (y_1 - y_2)^2 + z_2^2$$

and in vector form,

$$r_{12} = -x_1\hat{i} + (y_2 - y_1)\hat{j} + z_2\hat{k}$$

where the normal vector for plane 1 is $n_1 = \hat{k}$ and $n_2 = \hat{i}$. Using these, we find the β to be:

$$\begin{aligned} \cos\beta_1 &= \frac{n_1 \cdot r_{12}}{|n_1||r_{12}|} && = \frac{z_2}{\sqrt{x_1^2 + (y_1 - y_2)^2 + z_2}} \\ \cos\beta_2 &= \frac{n_2 \cdot r_{12}}{|n_2||r_{12}|} && = \frac{-x_1}{\sqrt{x_1^2 + (y_1 - y_2)^2 + z_2}} \end{aligned}$$

and the view factor is given as

$$F_{12} = \frac{1}{\pi L^2} \int_0^L \int_0^L \int_0^L \int_0^L \frac{-x_1 z_2}{\left(x_1^2 + (y_1 - y_2)^2 + z_2^2\right)^2} \, \mathrm{d}x_1 \, \mathrm{d}y_1 \, \mathrm{d}y_2 \, \mathrm{d}z_2$$

which yields approximately 0.2.

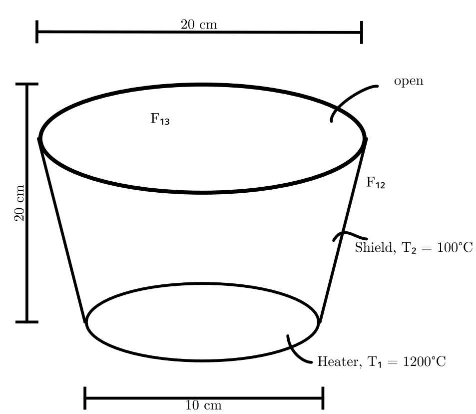

Figure 39: Example cone problem

In figure 39, we don't have a book value for the view factor of a cone. However, from the perspective of the bottom disc 1:

$$1 = F_{12} + F_{13}$$

and we do have a book falue for the top disc book factor. Therefore, we can calculate the view factor of the side cone using a simple subtraction. This enables us to find out how much heat is transferred to the shield, and how much goes to the open environment.

The above were black body problems. A gray body would not absorb all radiation, and therefore is slightly harder.

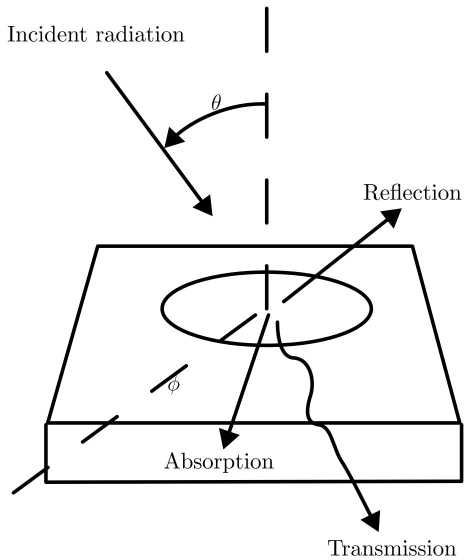

Figure 40: Gray body waves

Shown in figure 40, the transmission is the fraction of energy incident transmitted through the body

$$\tau_\lambda(T, \lambda, \theta, \phi)$$

and the reflected energy fraction is

$$\rho_\lambda(T, \lambda, \theta, \phi)$$

and that being absorbed is

$$\alpha_\lambda(T, \lambda, \theta, \phi).$$

The net must equal to 1:

$$\rho_\lambda + \alpha_\lambda + \tau_\lambda = 1.$$

If the body is isotropic (diffuse), then $\rho_\lambda, \alpha_\lambda, \tau_\lambda$ would not depend on the direction θ, ϕ. If the $\tau_\lambda = 0$, then the body can be considered opaque at that wavelength. It can be considered 'just opaque' if it does not allow transmission for a large range of wavelength, such as steel.

Kirchoff's Law says that the emittance at a wavelength λ and direction θ, ϕ is equal to the absorption at that direction and wavelength.

$$\varepsilon_\lambda(\theta, \phi) = \alpha_\lambda(\theta, \phi).$$

This is true at thermal equilibrium, which is almost always the case in this class. This is due to a simple law of thermodynamics: if the emission was greater than the absorption, then the body 'creating energy' if its temperature was still constant. For an isotropic body, the direction does not matter. Moreover, if the body is gray, then ε_λ does not depend on the wavelength λ. All these special cases gives us

$$\varepsilon = \alpha.$$

Notice that this is also true for black bodies, except that black bodies are restricted to $\varepsilon = \alpha = 1$. Also, a gray body may not necessarily always be diffuse.

For an opaque diffuse gray body:

$$\rho = 1 - \alpha = 1 - \varepsilon.$$

Heat transfer between parallel, gray, and opaque planes will have continuous reflections between each other, shown in figure 41. An emission from plane 1 is given as $\varepsilon_1 e_{b1}$, which gets reflected as

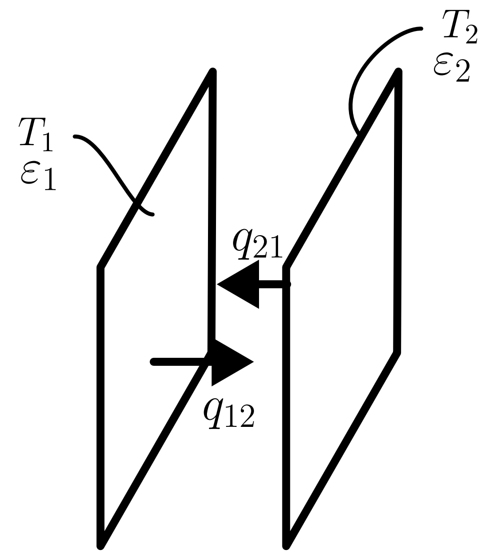

Figure 41: Gray body parallel planes

$\rho_2\varepsilon_1 e_{b1}$ off of plane 2, which gets reflected again from plane 1 as $\rho_1\rho_2\varepsilon_1 e_{b1}$. We can write this as an infinite series. The same is true for radiation initially coming from plane 2, $\varepsilon_2 e_{b2}$. The net radiation from 1 to 2 would be

$$\begin{aligned} q_{12,net} = \varepsilon_1 e_{b1}(1 + \rho_1\rho_2 + \rho_1^2\rho_2^2 + \dots) \\ + \rho_1\varepsilon_2 e_{b2}(1 + \rho_1\rho_2 + \dots) \\ - \varepsilon_2 e_{b2}(1 + \rho_1\rho_2 + \dots) \\ - \rho_2\varepsilon_1 e_{b1}(1 + \rho_1\rho_2 + \dots) \end{aligned}$$

which is simply

$$q_{12,net} = \frac{\varepsilon_1(e_{b1} - \rho_2 e_{b1}) - \varepsilon_2(e_{b2} - \rho_1 e_{b2})}{1 - \rho_1\rho_2}. \tag{193}$$

where the summation series was

$$\sum_{n=0}^{\infty}(\rho_1\rho_2)^n = \frac{1}{1 - \rho_1\rho_2}.$$

This can be re-arranged as

$$q_{12,net} = \frac{e_{b1} - e_{b2}}{\frac{1}{\varepsilon_2} + \frac{1}{\varepsilon_1} - 1} = \frac{\sigma(T_1^4 - T_2^4)}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}.$$

Notice that there is no view factor because these are infinite planes; however, there is a transfer factor:

$$\mathcal{F}_{12} = \frac{1}{\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - 1}. \tag{194}$$

The other way to do this problem is to consider a net radiation going in to the plane as H_1 and its reflected part as $\rho_1 H_1$. The outgoing radiation (emission and reflection) is known as the radiosity:

$$B_1 = \varepsilon_1 e_{b1} + \rho_1 H_1 \tag{195}$$

and is equal to 1 for this infinite plane. This also forces $B_1 = H_2$ and $B_2 = H_1$. This can be solved algebraically and does not require any infinite series. This method is more practical than the ray tracing method. The H is a flux, just like B. H is called the irradiance in the textbook. The resulting solution is

$$\begin{aligned} B_1 &= \frac{\varepsilon_1 e_{b1} + (1-\varepsilon_1)\varepsilon_2 e_{b2}}{1-(1-\varepsilon_1)(1-\varepsilon_2)} \\ B_2 &= \frac{\varepsilon_2 e_{b2} + (1-\varepsilon_2)\varepsilon_1 e_{b1}}{1-(1-\varepsilon_1)(1-\varepsilon_2)} \end{aligned}$$

The net flux is simply

$$q_{12,net} = B_1 - B_2.$$

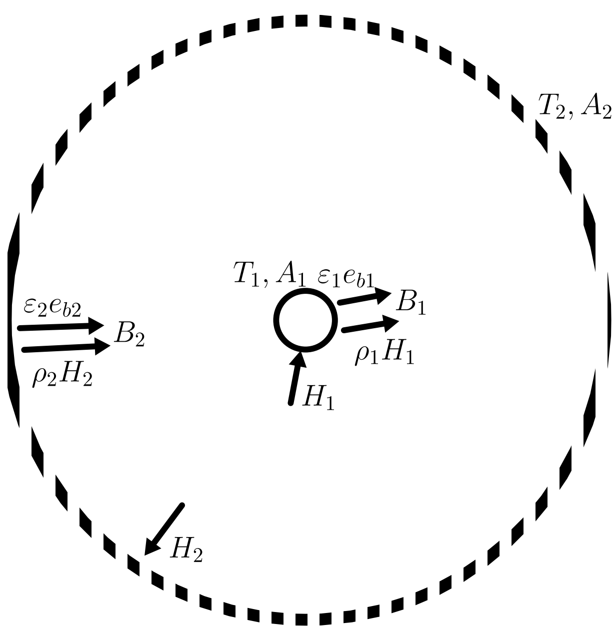

Figure 42: Concentric Gray Opaque Spheres

Shown in figure 42, the view factor is simply

$$F_{21} = \frac{r_1^2}{r_2^2} = \frac{A_1}{A_2}$$

and $F_{22} = 1 - \frac{A_1}{A_2}$. The algebraic equations are the same as the parallel planes, except that the irradiance is now

$$\begin{aligned} H_2 &= B_1 F_{21} + B_2 F_{22} \\ H_1 &= B_2 F_{12} = B_2. \end{aligned}$$

23 Lecture 23

Homework is due Thursday night, not Friday. The final exam will cover the following chapters: 6.1–8, 7.1–6, 8.1–5. Final is on Wednesday May 6th, 9 AM to 11 AM. We can have the textbook and our notes. It will also likely be two problems.

Gray sphere in a gray spherical enclosure will be the starter to harder problems. Gray means that the emissivity does not depend on λ wavelength. This is a continuation of figure 42. The black body radiation of the inner sphere is $e_{b1} = \sigma T_1^4$ and $e_{b2} = \sigma T_2^4$ for the outer. The radiosity is defined as

$$B_1 = \varepsilon_1 e_{b1} + (1-\varepsilon_1) H_1$$

and similarly for the outer sphere. The irradiance is given as

$$\begin{aligned} H_1 &= F_{12} B_2 = B_2 \\ H_2 &= F_{21} B_1 + F_{22} B_2. \end{aligned}$$

The view factors are known. These represent 4 equations and 4 unknowns. Combining the H into B equations, we get

$$\begin{aligned} B_1 &= \varepsilon_1 e_{b1} + (1-\varepsilon_1) B_2 \\ B_2 &= \varepsilon_2 e_{b2} + (1-\varepsilon_2)\left(\frac{A_1}{A_2} B_1 + \left(1 - \frac{A_1}{A_2}\right) B_2\right) \end{aligned}$$

and substituting these two together yields the solution

$$B_1 = \frac{e_{b1}\left(A_1\varepsilon_1(1-\varepsilon_1) + A_2\varepsilon_1\varepsilon_2\right) + e_{b2}\left(A_2\varepsilon_2 - A_2\varepsilon_1\varepsilon_2\right)}{A_1\varepsilon_1(1-\varepsilon_2) + A_2\varepsilon_2}. \tag{196}$$

and

$$H_1 = \frac{B_1 - \varepsilon_1 e_{b1}}{1-\varepsilon_1}.$$

Combining these two, we get the net heat flux that leaves the inner sphere:

$$\begin{aligned} q_{1,net} &= B_1 - H_1 \\ &= \frac{e_{b1} - e_{b2}}{\dfrac{1}{\varepsilon_1} + \dfrac{A_1}{A_2}\dfrac{1-\varepsilon_2}{\varepsilon_2}}. \end{aligned} \tag{197}$$

If we now supposed that the outer sphere is a black body, then it has $\varepsilon_2 = 1$ and also has no reflections. This would transform

$$q_{1,net} = \varepsilon_1 \sigma (T_1^4 - T_2^4)$$

which should be familar from chapter 1, where we had a body radiate to a black enclosure. The net heat is

$$Q_{1,net} = A_1 \varepsilon_1 \sigma (T_1^4 - T_2^4)$$

If we assumed that the enclosure is very large and not necessarily a black body, $A_2 \gg A_1$ and $\varepsilon_2 \neq 1$. This would still make the net heat similar to above. The radiation emitted from the inner sphere hardly ever returns. Additionally, the enclosure need not be spherical so long as the return back to the inner sphere is small.

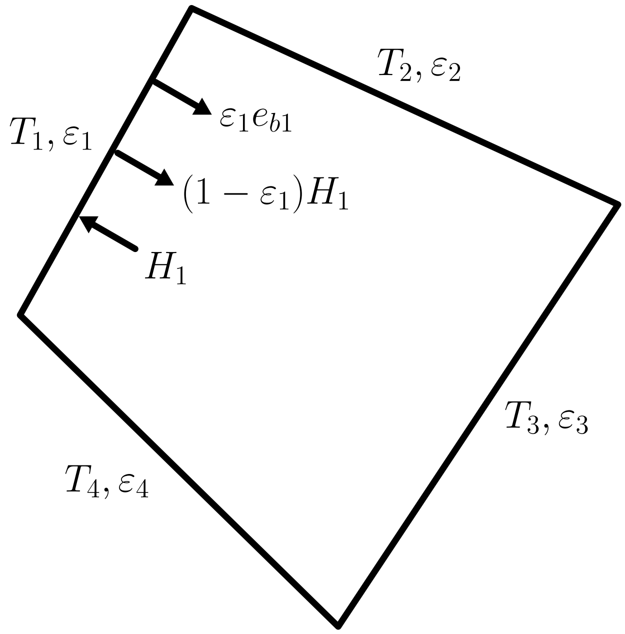

Figure 43: General Gray Enclosure

Assume that we have an arbitrary enclosure like in figure 43. Any surface can emit, receive, and reflect radiation. This has the equations

$$B_1 = \varepsilon_1 e_{b1} + (1-\varepsilon_1)H_1$$

and we also define the net heat this time

$$Q_{1,net} = A_1(B_1 - H_1) = \frac{\text{netheat}}{\text{time}} \text{ leaving } 1$$

which is

$$\begin{aligned} Q_{1,net} &= A_1\left(B_1 - \frac{B_1 - \varepsilon_1 e_{b1}}{1-\varepsilon_1}\right) \\ &= A_1\left(\frac{\cancel{B_1} - \varepsilon_1 B_1 - \cancel{B_1} + \varepsilon_1 e_{b1}}{1-\varepsilon_1}\right) \\ &= \frac{e_{b1} - B_1}{\frac{1-\varepsilon_1}{A_1\varepsilon_1}} \end{aligned} \tag{198}$$

and is written like a resistance equation. Equations for the other surfaces are similar. Additionally, note that we are making a restriction: no surface sees itself.

The irradiance is given as

$$H_1 = F_{12}B_2 + F_{13}B_3 + F_{14}B_4.$$

or simply

$$H_1 = -\frac{Q_{net}}{A_1} + B_1.$$

Notice that this can be re-arranged to

$$Q_{1,net} = A_1\left[\left(F_{12} + F_{13} + F_{14}\right)B_1 - \left(F_{12}B_2 + F_{13}B_3 + F_{14}B_4\right)\right]$$

where $F_{12} + F_{13} + F_{14} = 1$ because surface 1 does not see itself; this was the basis for not including it in the earlier equation for $Q_{1,net}$. Introducing it now enables us to perform the algebraic manipulation

$$Q_{1,net} = A_1\left[F_{12}(B_1 - B_2) + F_{13}(B_1 - B_3) + F_{14}(B_1 - B_4)\right]$$

which can now be written as a resistance by bringing the A_1 inside:

$$Q_{1,net} = \frac{B_1 - B_2}{\frac{1}{A_1F_{12}}} + \frac{B_1 - B_3}{\frac{1}{A_1F_{13}}} + \frac{B_1 - B_4}{\frac{1}{A_1F_{14}}} \tag{199}$$

and this is similar to the net heat for other surfaces. Notice that we have equations 198 and 199 which can now be related, along with the equations for the other surfaces. This is the general approach for any number of surfaces. Keep in mind that the surfaces cannot see themselves and are opaque and gray.

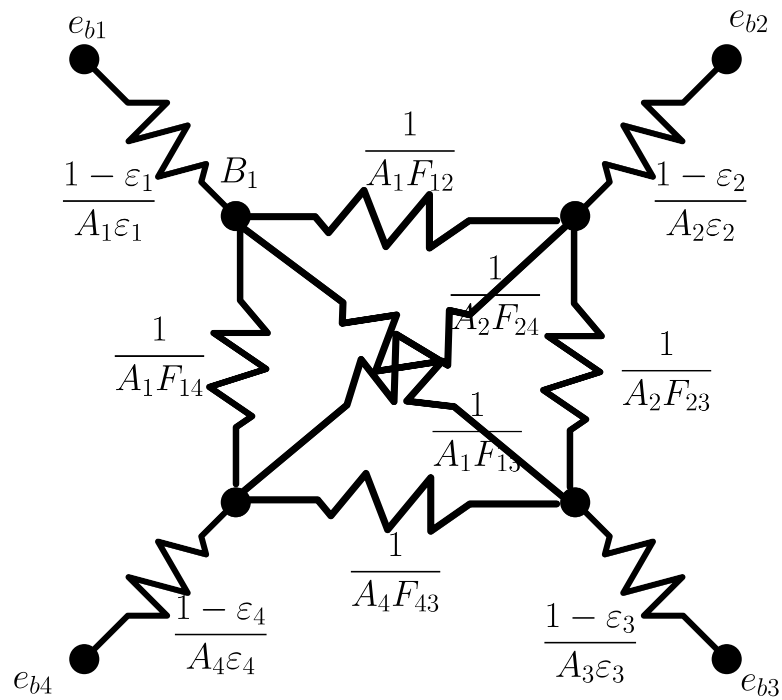

Figure 44: General Gray Enclosure Resistances

We can turn this into nodes. This is shown in figure 44 which shows the resistances: the blackbody radiation is like a battery here, with an internal resistance.

If the bodies are not gray, the equations and resistances simply have to be written for a particular wavelength λ. This is important for things like glass, which does not allow infrared light to pass, so special glass is needed for infrared lenses.

If a body is a good reflector, i.e. a shield, then the ϵ should be close to zero. In the circuit diagram, it is equivalent to having an open circuit. This is the case for aluminum foil.

If the frequency of the radiation is close to the frequency of the oscillation of the molecules, the radiation can go into the molecules; it counts as absorption. They typically absorb in only specific wavelength bands.

Questions