Introduction to Differential Equations
MATH 225

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Contents

1 August 31: Introductions

Hye-Won Kang is the professor for the course, MATH 225. She will post content on the BlackBoard page. Her email is hwkang@umbc.edu. Her notifications will be posted on BlackBoard.

The textbook used in the course is: *An Introduction to Differential Equations and Their Applications* by Stanley J. Farlow. It is recommended that we buy the paper copy version. Errors in the online copy of the book can be found here.

The syllabus will be posted before the weekend. We will have mostly lectures in the same format as today. We will have two exams: midterm and final. There exists 6 chapters, 1 through 6. Weekly quizzes and activities, and reading problems are part of the grading. The lowest 2 grades will be dropped from these; these contribute to 30% of our grade. The midterm contributes to 30%. The finals are 40%. Although we will have homework, it will neither be graded nor collected. We are permitted to come to office hours and ask how to obtain the solutions. The weekly activities may be in group or individual, multiple choice, etc. Exams will have multiple choice questions, long answer questions, etc.

We will need scientific calculators. The professor factors in weekly activities for the final letter grade. The quizzes will be taken during lecture. Since she taught the class in 2013 (or three years ago, misheard), she will provide sample practice equations.

List of Questions to Ask

1. Are we to bring the book to our classes?
2. Are computers allowed during lectures?

1.1 Differential Equations: What Are They?

DE commonly involve time, and other independent variables that are continuous: amount of resources, stock prices, etc. DE dependent on multiple variables are multivariable, and are a different consideration.

The instantaneous rate of change can be written as the derivative:

$$\frac{dy}{dt} = \frac{100}{y} \tag{1.1.1}$$

This has the solution:

$$y(t) = \sqrt{200t} \tag{1.1.2}$$

The astronomer Galileo Galilei found that, regardless of mass of objects, falling objects have the same acceleration (excluding friction). This would, therefore have the equation:

$$a(t) = g = v'(t) = \frac{dv(t)}{dt} = \frac{dv}{dt} = s''(t) = \frac{d^2s}{dt^2} \tag{1.1.3}$$

All the above are the same, simply different representations. Solving for that equation would yield the position function for a falling object:

$$s(t) = \frac{1}{2}gt^2 + C \tag{1.1.4}$$

1.2 Notations and Definitions

In DE, one (or multiple, if multivariable), is the independent variable, such as time. One variable is the dependent variable, such as y. This can be written as is or as a function of x: $y(x)$. The derivatives, therefore, are written as

$$\frac{dy}{dx} = y' \tag{1.2.1}$$

$$\frac{d^2y}{dx^2} = y'' \tag{1.2.2}$$

$$\vdots = \vdots \tag{1.2.3}$$

$$\frac{d^{n-1}y}{dx^{n-1}} = y^{(n)} \tag{1.2.4}$$

The general solution to DE is given by

$$F(x, y, y', y'', \cdots, y^{(n)}) = 0 \tag{1.2.5}$$

DE can be linear or non-linear. The nth order linear differential equation can be written as:

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \tag{1.2.6}$$

In essence, it is a series of differentials multiplied with a constant (which can be a variable). If $f(x) = 0$, the equation is said to be homogenous, otherwise it is non-homogenous. The 'order' of a differential equation is determined by the **highest** derivative order.

The following equation:

$$x^2 \cdot \frac{dy}{dx} + y^2 + 2y - 3 = 0 \tag{1.2.7}$$

is 1st-order, however, is non-linear due to the y^2 term. This is because, according to equation 1.2.6, the "constants" should be in respect to x, and there is only one y term.

Additionally, the problem 1.2.7 is non-homogenous due to the constant present '-3'. Note that moving the constant to the right yields $f(x) = 3 \neq 0$. Additionally, note that the y term is actually dependent on x: $y = y(x)$.

1.3 Problems

$$\frac{d^2y}{dx^2} + x^4y = \sin y + 5 \tag{1.3.1}$$

$$\frac{d^2y}{dx^2} + x^4y = \sin x + 5 \tag{1.3.2}$$

Both equations are 2nd-order. However, note that the term $\sin y$ is not among the variables for equation 1.3.1 present in definition 1.2.5, therefore it is non-linear. In contrast, the second equation is linear, because the left-hand-side is equal to some function of x: $f(x)$.

Homogenuity refers to whether $f(x) = 0$.

Linearity refers to equation definition 1.2.6.

Additionally, even if $y = x$, equation 1.3.1 will still be non-linear, because **by definition**, we said '$f(x,)$'

Problem: Constructing a first-order linear homogenous differential equation.

My Solution:

$$a\dot{\theta} + b\theta = 0 \ni a, b \in \mathbb{R} \tag{1.3.3}$$

Three Important Definitions

$$\text{DE:} \quad F(x, y, y', y'', \cdots, y^{(n)}) = 0 \tag{1.3.4}$$

$$\text{Linear:} \quad a_0(x)y^{(n)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \tag{1.3.5}$$

$$\text{Homogenuous:} \quad f(x) = 0 \tag{1.3.6}$$

2 September 7: Solutions to Differential Equations

A list of suggested problems will be provided online. The quizzes will begin after September 14, 2022. We are required to use a scientific calculator during exams. In the previous class, we went over DE, linear, homogenous, 2nd order, homogenous, etc. equations.

Variables can be different in equations:

$$\frac{d^2v}{dt^2} = t^2v$$

The independent variable here is t, and v is the dependent.

Important Note to Professor: The link in the syllabus for errors has a different tilde character; it possesses the unicode `U+223C`, whereas we need `U+007E`.

The equation above is homogenous because we can rewrite the equation as:

$$\ddot{v} - t^2v = 0$$

which matches equation definition 1.3.6. If an equation is a linear differential equation, we can classify homogenuity. However, homogenuity cannot be classified for nonlinear equations.

Example: Finding Solutions of Differential Equations

Given $\ddot{s} = g$, find s. Using integration, we find that:

$$\begin{aligned} \ddot{s} &= g \\ \dot{s} &= gt + C_1 \\ s &= \frac{1}{2}gt^2 + C_1t + C_2 \end{aligned} \tag{2.0.1}$$

We can verify by computing derivatives of s.

For a circle centered at the origin, the equation can be:

$$\frac{dy}{dx} = -\frac{x}{y}$$

However, this is non-linear according to equation 1.3.5. Question: Will

$$x^2 + y^2 = 4 \tag{2.0.2}$$

satisfy the previous non-linear equation? Solution:

$$2x + 2y\dot{y} = 0 \tag{2.0.3}$$

$$\dot{y} = -\frac{x}{y} \tag{2.0.4}$$

Step 2.0.3 was obtained by taking the *implicit derivative* of 2.0.2. Rearranging the answer shows that indeed that that is a solution.

We can further verify:

$$y = \sqrt{4 - x^2} \tag{2.0.5}$$

$$\dot{y} = \frac{1}{2\sqrt{4 - x^2}} \cdot -2x$$

$$= -\frac{x}{\sqrt{4 - x^2}} \tag{2.0.6}$$

$$= -\frac{x}{y} \tag{2.0.7}$$

Since this solution matches 2.0.4. However, note that when the solution was obtained implicitly, y could be both positive and negative. In contrast, for the *explicit* solution, $y \geq 0$. They are not an exact match, but we will not focus on that too much. We could make it a match by saying that:

$$\frac{dy}{dx} = \frac{d}{dx}\sqrt{4 - x^2} \iff y \geq 0 \tag{2.0.8}$$

2.1 Initial Value Problem

Referring back to 2.0.1, we can set the initial conditions $s = 5, t = 0$ and $\dot{s} = 0, t = 0$, or any other initial conditions.

When solving any nth order DE, we need n initial conditions:

$$\begin{aligned} y(0) &= y_0 \\ y'(0) &= y_1 \\ \vdots &= \vdots \\ y^{(n-1)}(0) &= y_{n-1} \end{aligned}$$

For the quiz next week, we may have multiple choice, short answers, and long answers. It will be about 15 minutes. We are highly encouraged to read *Picard's Theorem*, which is Theorem 1.1 in the book. The Theorem discusses:

1. The existence of solution
2. The uniqueness of solution

2.2 Solving First-Order Linear Differential Equations

$$\frac{dy}{dx} - y = 0 \tag{2.2.1}$$

The above equation matches all the definitions 1.3.4 to 1.3.6. In particular, first-order linear DE have the format:

$$\dot{y} + p(x) \cdot y = f(x) \tag{2.2.2}$$

Note that $\dot{y}$ does not have the constant because the constant has been multiplied with the entire equation. It is homogenous if $f(x) = 0$.

2.2.1 Integrating Factor Method

$$\dot{y} + 3y = e^{-2x} \mid y(0) = 5 \tag{2.2.3}$$

$$\text{Integrating factor: } = e^{3x} \tag{2.2.4}$$

$$\Rightarrow e^{3x}(\dot{y} + 3y) = e^{x} \tag{2.2.5}$$

$$\text{Since } \frac{d}{dx}\left(e^{3x} \cdot y\right) = \frac{d}{dx}(e^{3x}) \cdot y + e^{3x} \cdot \frac{d}{dx}(y) \tag{2.2.6}$$

$$\implies \frac{d}{dx}\left(e^{3x} \cdot y\right) = e^{x} \tag{2.2.7}$$

$$e^{3x}y = \int e^{x}\, dx \tag{2.2.8}$$

$$= e^{x} + C \tag{2.2.9}$$

$$\therefore y = e^{-2x} + C \cdot e^{-3x} \tag{2.2.10}$$

Step 2.2.6 is used to rewrite the step prior.
For next class, try:

$$\dot{y} + 3y = \sin x \mid y(0) = 1 \tag{2.2.11}$$

3 September 12: Solutions to Differential Equations

Last time, we talked about the integrating factor method, where we multiply an entire equation with some e^{ax}. This is done so that we can use derivative to obtain solution:

$$\frac{d}{dx}(e^{ax}y) = e^{ax}f(x) \tag{3.0.1}$$

$$e^{ax}y = \int e^{ax}f(x)\,dx + C \tag{3.0.2}$$

$$y = e^{-ax}\int e^{ax}f(x)\,dx + Ce^{-ax} \tag{3.0.3}$$

We will have a **quiz on Wednesday**. We will cover the material that we went over previously, up to integrating factor method.

3.1 Integrating Factor Method

In general, we use the multiplier $\mu(x) : e^{\int \mu(x)\,dx}$:

$$\dot{y} + \mu(x)y = f(x)$$
$$e^{\int \mu(x)\,dx}(\dot{y} + \mu(x)y) = \frac{d}{dx}\left(e^{\int \mu\,dx}y\right)$$
$$e^{\int \mu(x)\,dx}(\dot{y} + \mu(x)y) = e^{\int \mu(x)\,dx}f(x)$$
$$\frac{d}{dx}\left(e^{\int \mu(x)\,dx}y\right) = e^{\int \mu(x)\,dx}f(x)$$

The above derivation can also be found in the book. As an example problem:

$$\dot{y} + 3y = \sin x \mid y(0) = 1 \tag{3.1.1}$$

$$\mu(x) = 3x \tag{3.1.2}$$

$$\frac{d}{dx}\left(e^{3x}y\right) = e^{3x}\sin x \tag{3.1.3}$$

$$e^{3x}y = \int e^{3x}\sin x\,dx + C \tag{3.1.4}$$

$$\int e^{3x}\sin x\,dx = uv - \int uv' \text{ Integration by Parts} \tag{3.1.5}$$

$$= \frac{1}{3}e^{3x}\sin x - \int \frac{1}{3}e^{3x}\cos x\,dx \tag{3.1.6}$$

$$= \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\int e^{3x}\cos x\,dx \tag{3.1.7}$$

$$= \frac{1}{3}e^{3x}\sin x - \frac{1}{3}\left[\frac{1}{3}e^{3x}\cos x - \frac{1}{3}\int e^{3x}(-\sin x)\,dx\right] \tag{3.1.8}$$

$$= \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x - \frac{1}{9}\int e^{3x}\sin x\,dx \tag{3.1.9}$$

$$\frac{10}{9}\int e^{3x}\sin x\,dx = \frac{1}{3}e^{3x}\sin x - \frac{1}{9}e^{3x}\cos x \tag{3.1.10}$$

$$\int e^{3x}\sin x\,dx = \frac{3}{10}e^{3x}\sin x - \frac{1}{10}e^{3x}\cos x \tag{3.1.11}$$

$$\Rightarrow e^{3x}y = \frac{3}{10}e^{3x}\sin x - \frac{1}{10}e^{3x}\cos x + C \tag{3.1.12}$$

$$y = \frac{3}{10}\sin x - \frac{1}{10}\cos x + Ce^{-3x} \tag{3.1.13}$$

$$y(0) = \frac{3}{10}\sin 0 - \frac{1}{10}\cos 0 + Ce^{0} \tag{3.1.14}$$

$$\implies C = \frac{11}{10} \tag{3.1.15}$$

Now, what is the integrating factor, and $\mu(x)$ for the following?

$$\dot{y} + 2xy = x \mid y(1) = 1 \tag{3.1.16}$$

The integrating factor, according to me, should be:

$$\mu(x) = 2x \implies e^{\int \mu(x)\,dx} = e^{x^2}$$

If there is any coefficient in front of the $\dot{y}$ term, we can divide the whole equation with *that* coefficient. We can verify that our solution is correct by plugging it back into the problem statement.

Try at home: Using the integrating factor method, solve the following:

$$\dot{y} + 2xy = x^3 \mid y(1) = 1 \tag{3.1.17}$$

3.2 Separation of Variables

First, using the integrating factor method, the solution for the following:

$$\begin{aligned} \dot{y} - 2y &= 0 \\ e^{-2x}(\dot{y} - 2y) &= 0 \\ \frac{d}{dx}\left(e^{-2x}y\right) &= 0 \\ e^{-2x}y &= 0 + C \\ y &= Ce^{2x} \end{aligned}$$

The above solution is actually the same equation as linear growth. In separation of variables:

$$\frac{dy}{dx} = 2y \tag{3.2.1}$$

$$\frac{1}{y}\,dy = 2\,dx \tag{3.2.2}$$

$$\int \frac{1}{y} dy = \int 2\,dx + C \tag{3.2.3}$$

$$\ln|y| = 2x + C \tag{3.2.4}$$

$$|y| = e^{2x+C} \tag{3.2.5}$$

$$y = \pm e^{2x+C} \tag{3.2.6}$$

In general, for:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \tag{3.2.7}$$

$$g(y)\,dy = f(x)\,dx \tag{3.2.8}$$

$$\int g(y)\,dy = \int f(x)\,dx + C \tag{3.2.9}$$

We should think about the following examples at home, and whether they can be solved via integrating factor method and separation of variables.

$$\frac{dy}{dx} = y^2 x \tag{3.2.10}$$

$$\frac{dy}{dx} = y^2 x + x \tag{3.2.11}$$

4 September 14: Picard's Theorem and Growth-Decay

Picard's Theorem

$$\text{If } \frac{dy}{dx} = f(x,y), \text{ and} \tag{4.0.1}$$

$$f(x,y) \text{ is continuous, and} \tag{4.0.2}$$

$$\frac{\partial f}{\partial y} \text{ is continuous,} \tag{4.0.3}$$

$$\exists \text{ a unique solution over the domain } D \tag{4.0.4}$$

Growth equation is given by:

$$\frac{dy}{dx} = ky \mid k > 0 \tag{4.0.5}$$

The k indicates the growth rate. $k = 2$ for example, could indicate population doubling. This is applicable to compound interests too. Solution to the equation 4.0.5 is known as the growth curve:

$$y = Ce^{kx} \tag{4.0.6}$$

Whether an equation is a solution to a DE can be determined by computing the derivative of the 'solution' (1), and inserting the solution into the DE (2); if (1) and (2) matches, then the 'solution' is indeed the solution.

Similarly, the decay equation is given by

$$\frac{dy}{dx} = -ky \mid k > 0 \tag{4.0.7}$$

and the solution is given by

$$y = Ce^{-kx} \tag{4.0.8}$$

Example Problems

Solving a growth equation using integrating factor method:

$$\begin{aligned} \dot{y} &= 10y \\ \frac{d}{dx}\left(e^{-10x}y\right) &= 0 \\ e^{-10x}y &= C \\ y &= Ce^{10x} \end{aligned}$$

Solving a growth equation using separation of variables:

$$\begin{aligned}
\frac{dy}{dx} &= 10y \\
\int \frac{1}{10y} dy &= \int 1\, dx \\
\frac{1}{10} \ln|y| &= x + C \\
|y| &= e^{10x+C} \\
y &= \pm C e^{10x}
\end{aligned}$$

Half-lives can be computed using decay curves. For example, for Cesium-137 with half life of 30 years. If x is the amount of time in years, and y is the amount, then $y = \frac{1}{2}y_0,\ x = 30$. To find k:

$$\begin{aligned}
y(30) &= Ce^{-30k} \\
&= \frac{C}{2} \\
\Rightarrow e^{-30k} &= \frac{1}{2} \\
-30k &= \ln 0.5 \\
k &= \frac{\ln 0.5}{-30} \\
&\approx 0.0231
\end{aligned}$$

To find when the sample will be 1% of original value:

$$\begin{aligned}
\frac{C}{100} &= Ce^{-kx} \text{ (we know } k) \\
\ln 0.01 &= -kx \\
x &= -\frac{\ln 0.01}{\frac{\ln 0.5}{-30}} \\
&\approx 199 \text{ yrs}
\end{aligned}$$

Compound interest of $r = 3\%$ per annum, with initial amount of \$1000, can be used in DE . After 10 years, the amount will be:

$$S(t) = S_0(1 + \frac{r}{n})^{nt}$$

where n is after how many years the interest is applied. Note that:

$$\lim_{n\to\infty} \left(1 + \frac{A}{n}\right)^{\frac{n}{A}} = e \tag{4.0.9}$$

This can be translated to equation 4.0.6:

$$S(t) = S_0 e^{rt} \tag{4.0.10}$$

We will have \$5000 when time is:

$$\begin{aligned}
\$5000 &= \$1000e^{0.05t} \\
5 &= e^{0.05t} \\
0.05t &= \ln 5 \\
t &= \frac{\ln 5}{0.05} \\
&= 33 \text{ yrs}
\end{aligned}$$

Annuity is regular deposits. When using annuity, the equation changes.

5 September 28: Growth-Decay, Numerical Approximation, and Second-Order Differential Equations

Equation for heating and cooling is given by

$$\frac{dT}{dt} = -k(T - M) \mid k > 0 \tag{5.0.1}$$

where T is temperature of object and M is temperature of surrounding. Equation for mixing given by

$$\frac{dQ}{dt} = \text{in} - \text{out} \tag{5.0.2}$$

Euler's Method is given by the equation

$$y_{n+1} = y_n + h \cdot f'(x_n) \tag{5.0.3}$$

where h is the step size.

Discretization error (truncation error) decreases as step size decreases, since prediction is more accurate. Round off error increases as step size decreases, since more error adds up. The total error is the summation of these two errors. The optimum step size is s.t. the total error is minimized. The maximum error possible is given by

$$\frac{h^2}{2!}\ddot{y}_n = \mathcal{O}(h^2) \tag{5.0.4}$$

This comes from the Taylor Series Expansion formula.

In first-order linear equations, we used equation 2.2.2; it was solvable using integrating factor method and separation of variables. This was applicable in growth/decay equations, interests, half life, heating/cooling, mixing, etc.

Second order linear equations are given in the general form

$$a_0(x) \cdot \ddot{y} + a_1(x) \cdot \dot{y} + a_2(x) \cdot y = g(x) \mid a_0(x) \neq 0 \tag{5.0.5}$$

can be re-written as

$$\ddot{y} + p(x)\dot{y} + q(x)y = f(x) \tag{5.0.6}$$

Picard's Theorem does not work in this case. Integrating factor method does not work either.

6 October Online: General Solutions of Homogenous 2nd-Order DE

Lectures were recorded and posted online.

When variables in front of ys is constant, the linear differential equation such as 5.0.5 is said to have constant coefficients. Otherwise, if it has e.g. x, then it is said to have variable coefficients.

When 5.0.6 is *homogenous*, then we can assume that y_1 and y_2 are solutions of the equation, as well as c_1y_1 and c_2y_2.

If we are given an initial condition for $\ddot{y}$, we still require $y(0)$ and $\dot{y}(0)$ since integrating $\ddot{y}$ yields unknown constants.

Example 6.1. *Solving:*

$$\begin{aligned}
\ddot{y} &= x + 1 \,|\, y(0) = 0,\ \dot{y}(0) = 1 \\
\Rightarrow \dot{y} &= \frac{x^2}{2} + x + c_1 \\
\Rightarrow y &= \frac{x^3}{6} + \frac{x^2}{2} + c_1 x + c_2 \\
\dot{y}(0) \implies \dot{y} &= \frac{x^2}{2} + x + 1 \\
y(0) \implies y &= \frac{x^3}{6} + \frac{x^2}{2} + x
\end{aligned}$$

Picard's Theorem worked for first-order DE , in 4. Something else exists for second-order.

Existence and Uniqueness of Solution for 2nd-Order DE

For the second order equation 5.0.6 with

$$\begin{bmatrix} y(x_0) = y_0 \\ \dot{y}(x_0) = y_1 \end{bmatrix}$$

If the coefficients $p(x)$, $q(x)$, and $f(x)$ are continuous for an interval $x \in (a, b)$, and $x_0 \in (a, b)$, then $\exists$ a *unique* solution satisfying the initial condition on $x \in (a, b)$.

Example 6.2. *For an equation*

$$\ddot{y} + \frac{1}{x-1} y = 3$$

our $p(x) = 0$, $q(x) = \frac{1}{x-1}$, *and* $f(x) = 3$. *Additionally, if* $y(0) = 0$ *and* $\dot{y}(0) = 1$*: on the interval* $x \in (-1, 1)$, $\exists$ *a unique solution on the interval, according to definition 6.*

Next, we will show that *all* solutions of homogenous 5.0.6 can be written as *the linear combination of two linearly independent solutions* 6.0.1, where cs are constants:

$$y = c_1 y_1 + c_2 y_2 \tag{6.0.1}$$

This comes from linear algebra, where vectors v_1 and v_2 are linearly independent if

$$\exists\, k_1 v_1 + k_2 v_2 = 0 \ni k_1 \neq 0,\ k_2 \neq 0 \tag{6.0.2}$$

As an example, $\sin x$ and $\cos x$ are linearly independent. In contrast, since $\sin 2x$ and $\sin x \cos x$ are the same, they are linearly *dependent.* x and $5x^2$ are linearly independent, while x and $5x$ are linearly dependent.

To determine if a function is linearly independent, we can conduct the Wronskian test:

Wronskian Test for Linear Independence

For a homogenous 2nd-order DE

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0$$

where $p(x)$ and $q(x)$ are continuous on $x \in (a, b)$, then the following statements are equivalent:

1. y_1 and y_2 are linearly independent solutions on $x \in (a, b)$
2. $W[y_1, y_2](x) \neq 0\, \forall \in (a, b)$
3. $W[y_1, y_2](x) \neq 0$ for at least one $x \in (a, b)$

where

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix} = y_1 \dot{y}_2 - y_2 \dot{y}_1 \tag{6.0.3}$$

Example 6.3. *If* $\sin 2x$ *and* $\cos 2x$ *are solutions of*

$$\ddot{y} + 4y = 0\, \forall\, x \in (-\infty, \infty)$$

then substituting with 6.0.1, we obtain

$$-4 \sin 2x + 4 \sin 2x = 0$$

and

$$W(\sin 2x, \cos 2x) = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix} = -2 \neq 0\, \forall\, x \in (-\infty, \infty)$$

Therefore, by definition 6, $\sin 2x$ *and* $\cos 2x$ *are linearly independent solutions for the above differential equation* $\forall\, x \in (-\infty, \infty)$.

If equation for 2nd-order DE is non-homogenous, then the y_1 and y_2 may not be linearly independent solutions of the equation.

All these steps were done in order to find *all* solutions for homogenous 2nd-order DE solutions:

Solution for Homogenous Differential Equation

Let y_1 and y_2 be two *linearly independent* solutions of

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0 \,\forall\, x \in (a, b)$$

Then *any solution* Y can be expressed as

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) \,\forall\, x \in (a, b)$$

where c_1 and c_2 are uniquely decided.

Proof:

$$c_1 y_1(x_0) + c_2 y_2(x_0) = Y(x_0) \tag{6.0.4}$$

$$c_1 \dot{y}_1(x_0) + c_2 \dot{y}_2(x_0) = \dot{Y}(x_0) \tag{6.0.5}$$

and then using elimination of both equations (like simultaneous equation):

$$c_1(y_1(x_0) \cdot \dot{y}_2(x_0) - \dot{y}_1(x_0) \cdot y_2(x_0)) = Y(x_0) \cdot \dot{y}_2(x_0) - \dot{Y}(x_0) \cdot y_2(x_0) \tag{6.0.6}$$

c_1 is decided uniquely if:

$$y_1(x_0) \cdot \dot{y}_2(x_0) - \dot{y}_1(x_0) \cdot y_2(x_0) \neq 0$$

but noticed that that is equivalent to the Wronskian $W[y_1, y_2](x_0) \neq 0$, and that is true because y_1 and y_2 are linearly independent. Similarly, c_2 is decided uniquely.

Fundamental and General Solutions

2 linearly independent solutions y_1 and y_2 of

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0$$

y_1 and y_2 are *a fundamental set of solutions*. Meanwhile, $c_1 y_1 + c_2 y_2 \,\forall\, c_1, c_2$ are the *general* solutions.

Questions to Ask

1. In definition 6, is 3 actually equivalent to 1?
2. Is it $W[]$ or $W()$?

3. In 47:08 of Monday's video, you stated that the Wronskian test (to be precise, y_1 and y_2 being linearly independent solutions) may not work, but wrote on the slides that it does not work. Which one is it?

7 October 10: Midterm Review

We will have an exam on this Wednesday. The entire class period would be reserved. For practice, it is best to review:

1. Quizzes
2. Lecture notes
3. HW problems

The exam will have 10 to 15 problems. Multiple choice requires no work.

Important things to note are:

1. Differential equation format.
$$F(x, y, y', y'', \cdots, y^{(n)}) = 0$$
2. Dependent and independent variables.
3. Order of differential equations, which involves questions about linearity, and if linear, homogenuity.
   (a) If 1st order, need to know integrating factor method and separation of variables method. Particularly if $p(x)$ is any of the following: x^n, e^{ax}, xe^{ax}, or $\sin x$
   (b) If 2nd order, its form.
4. Creating our own linear DE of any order.
5. Verifying solutions of DE with given conditions. In this case, make sure to compute both the left side and the right side in the same manner.
6. Explicit versus implicit solutions.
7. Initial value problems. For 1st-order, only $y(x_0)$ is needed, while 2nd-order needs both $y(x_0) = y_0$ and $\dot{y}(x_0) = y_1$. Note that both need to be at x_0.
8. Picard's Theorem: Existence & Uniqueness of Solution.
9. Solving first-order linear DE .
10. Using integrating factor method. Note cases where it may not work.
11. Using separation of variables method.
12. When each method works or does not.Note that separation of variables method can work even if the equation is non-linear.
13. Growth and decay equations.

14. Solving half life questions. We can go straight from $\frac{dy}{dx} = -ky$ to $y(x_0) = y_0 e^{-kt}$.
15. Continuous compound interest, $\frac{dS}{dt} = r \cdot S$, and $S(t) = S_0 e^{rt}$.
16. Compound interest with annuity $\frac{dS}{dt} = rS + A$.
17. Mixture phenomenon, where $Q(t)$ is the amount. Three cases are:
    (a) Flow rate in and out are the same.
    (b) Rate in less than rate out.
    (c) Rate in greater than rate out.
18. Setting the equations up for mixtures (previous). In the latter two, the volume is not constant. Therefore, the concentration for the out will be $\frac{Q}{V_0 + \dot{V}t}$. In the second case, the time has to be constrained since at one point, no volume would be left.
19. Cooling and heating phenomenon:
$$\frac{dT}{dt} = -k(T - M) \ni k > 0$$
    Heating if $T < M$, and cooling if $T > M$.
20. Newton's second law for falling objects.
$$m \cdot \frac{dv}{dt} = mg - kv \tag{7.0.1}$$
    where k is some resistance.
21. Directional fields, using the slope.
22. Using Euler's method: $y_{n+1} = y_n + h \cdot f(x_n, y_n)$
23. Errors. Discretization error decreases as step size decreases, but round off error increases.
24. Second order DE .
25. Integrating factor method does not work for 2nd order DE .
26. Principle of superposition: if y_1 and y_2 are solutions of a 2nd order DE , then the linear combinations of y_1 and y_2 are also solutions of the equations. This only works for *homogenous* DE .
27. Initial value problems for 2nd-order DE .
28. Existence and uniqueness of solution, refer to 6.
29. Determining linear independence of functions.
30. Computing the Wronskian.
31. Using the Wronskian for linear independence.

8 October 17: Reduction of Order and Constant Coefficients

We will learn how to solve equations of the form 5.0.6. For the homogenous 2nd-order equation,

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0 \tag{8.0.1}$$

if we know one solution $y_1(x)$ we can write $y_2(x) = \nu y_1(x)$ and find $\nu(x)$. y_2 is another solution to the equation, that is linearly independent.

Example 8.1. *Given that the solution of $\ddot{y} + \dot{y} - 2y = 0$ is $y_1(x) = e^x$, find another linearly independent solution $y_2(x)$. Solution:*

$$\begin{aligned}
y_2(x) &= \nu(x)e^x \\
\implies \ddot{y}_2 + \dot{y}_2 - 2y &= \frac{d^2}{dx^2}(\nu(x)e^x) + \frac{d}{dx}(\nu(x)e^x) - 2(\nu(x)e^x) \\
&= \frac{d}{dx}(\nu' e^x + \nu e^x) + (\nu' e^x + \nu e^x) - 2\nu e^x \\
&= \nu'' e^x + \nu' e^x + \nu' e^x + \nu e^x + \nu' e^x + \nu e^x - 2\nu e^x \\
&= e^x\left(\nu'' + 3\nu'\right) \ \textit{reduction of order} \\
w(x) &= \nu'(x) \\
\implies \nu'' + 3\nu' &= w'(x) + 3w(x) \\
\frac{dw}{dx} &= -3w \\
\frac{1}{w}dw &= -3\,dx \\
\ln w &= -3x + C \\
\Rightarrow w(x) &= Ce^{-3x} \\
\therefore \nu(x) &= \int Ce^{-3x}\,dx \\
&= C_1 e^{-3x} + C_2
\end{aligned}$$

Reduction of Order Method

If a non-zero solution of $y_1(x)$ is known for

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0$$

is known, then a second linearly independent solution $y_2(x)$ can be determined by substituting $y_2(x) = \nu(x)y_1(x)$, where

$$\nu(x) = \int \frac{e^{-\int p(s)ds}}{y_1^2(r)}\,dr \tag{8.0.2}$$

If we do not have any solution for a given solution, we can assume that $y(x) = e^{mx}$. Using this method, we would get for $\ddot{y} + \dot{y} - 2y = 0$:

$$\begin{aligned}
\frac{d^2}{dx^2}(e^{mx}) + \frac{d}{dx}(e^{mx}) - 2(e^{mx}) &= 0 \\
\Rightarrow e^{mx}\left(m^2 + m - 2\right) &= 0 \\
(m-1)(m+2) &= 0 \\
\Rightarrow m &= 1 \\
\text{or } m &= -2 \\
\therefore y_1(x) &= e^x \\
\text{and } y_2(x) &= e^{-2x}
\end{aligned}$$

Exercise assigned to us:

Example 8.2. *Find a general solution for* $4\ddot{y} + \dot{y} - 5y = 0$*:*

$$\begin{aligned}
4(e^{mx})'' + (e^{mx})' - 5(e^{mx}) &= 0 \\
\Rightarrow 4m^2e^x + me^x - 5e^{mx} &= 0 \\
e^{mx}\left(4m^2 + m - 5\right) &= 0
\end{aligned}$$

$$\begin{aligned}
m &= \frac{-1 \pm \sqrt{1^2 - 4 \cdot 4 \cdot -5}}{2 \cdot 4} \\
&= \frac{-1+9}{8}, \frac{-1-9}{8} \\
&= 1, -\frac{5}{4}
\end{aligned}$$

Hence, our general solution is:

$$y(x) = c_1e^x + c_2e^{-5x/4}$$

According to the above example(s):

Homogenous Equations with Constant Coefficients

For a 2nd-order homogenous DE with constant coefficients:

$$a\ddot{y} + b\dot{y} + cy = 0$$

can be re-written as

$$\begin{aligned}
am^2 + bm + c = 0 \ni y(x) = e^{mx} &\text{ where if} \\
D > 0 \, \exists \, m \ni m = r_1, r_2 \\
D = 0, \exists \, m = r \\
D < 0, \exists \, m = p \pm qi
\end{aligned}$$

Although 8.0.2 integrates in terms of s and r, it is to distinguish that that y and the e component are to be integrated separately, and then integrated together. They both, for the final integration, should hae the same variable.

Example 8.3. *Verifying that $y(x) = c_1e^{2x}+c_2xe^{2x}$ is a solution of $\ddot{y}-4\dot{y}+4y = 0$: Plugging in y_1 and y_2 individually to the equation, we get*

$$\begin{aligned} y_1 : \quad & 4e^{2x} - 8e^{2x} + 4e^{2x} = 0 \\ y_2 : \quad & (2e^{2x} + 2e^{2x} + 4xe^{2x}) - 4(e^{2x} + 2xe^{2x}) + 4xe^{2x} = 0 \end{aligned}$$

Therefore, y_1 and y_2 are solutions to the equation. Additionally,

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} \\ &= e^{2x}(e^{2x} + 2xe^{2x}) - xe^{2x} \cdot 2e^{2x} \\ &= e^{4x} + 2xe^{4x} - 2xe^{4x} \\ &= e^{4x} \neq 0 \, \forall \, x \in \mathbb{R} \end{aligned}$$

Therefore, $y_1 = e^{2x}$ and $y_2 = xe^{2x}$ are linearly independent solutions of $y(x)$, according to theorem 6.

9 October 19: Non-homogenous Equations

The midterm results were returned; the author is pleased. Question 6 had the most amount of errors. A was obviously true. C and D implied each other, therefore unique solution would not exist; hence B would be false. Most students did good in the midterm.

Previously we looked at homogenous equations with constant coefficients. When we have m to be a *complex* number (real part and complex part), we can write $y(x) = c_1e^{mx} + c_2e^{mx}\sin x \ni m \in \mathbb{R}$ for complex numbers:

$$y(x) = c_1e^{m_Rx}\cos(m_Cx) + c_2e^{m_Rx}\sin(m_Cx) \Big| \, m_R + m_Ci = m \in \mathbb{C} \tag{9.0.1}$$

This is given by the Euler's formula for complex numbers:

$$e^{ix} = \cos x + i \sin x \tag{9.0.2}$$

What if the equation is non-homogenous? As an example:

Example 9.1. *Will $y_p = x^3$ be a solution of $\ddot{y} - y = -x^3 + 6x$? Using 8:*

$$\begin{aligned} & \ddot{y} - y = -y + \ddot{y} \\ \implies & \ddot{y} - y = 0 \\ \Rrightarrow & m^2 - 1 = 0 \\ & m = \pm 1 \\ \implies & y_h(x) = c_1e^x + c_2e^{-x} \end{aligned}$$

We can prove:

$$\begin{aligned} Y(x) &= y_h(x) + y_p(x) \\ Y''(x) - Y(x) &= (y_h(x) + y_p(x))'' - (y_h(x) + y_p(x)) \\ &= (y_h''(x) - y_h(x)) + \left(y_p''(x) - y_p(x)\right) \\ &= 0 \end{aligned}$$

Superposition Principle for Non-Homogenous Equations

Given an input-output system

$$\ddot{y} + p(x)\dot{y} + q(x)y = f(x) \tag{9.0.3}$$

where $f(x)$ is the input and $y(x)$ is the output, we can say that it has the properties:

1. **Scalar Property** If $y_1(x)$ is a solution of 9.0.3, and we use the input $k \cdot f(x)$, then the output will be $k \cdot y_1(x)$; that is, $k \cdot y_1(x)$ is a solution of
$$\ddot{y} + p(x)\dot{y} + q(x)y = k \cdot f(x) \tag{9.0.4}$$
2. **Additive Property** If $y_1(x)$ is a solution of the input $f_1(x)$, and $y_2(x)$ is a solution of the input $f_2(x)$, then an input of $f_1(x)+f_2(x)$ will yield a solution of $y_1(x) + y_2(x)$.

Example 9.2. *Given $\ddot{y} + \dot{y} = x^2 + 2x$ with $y_1(x) = \frac{1}{3}x^3$, and $\ddot{y} + \dot{y} = 6e^{2x}$ with $y_2(x) = e^{2x}$, then what is the solution of the following?*

$$\ddot{y} + \dot{y} = -e^{2x} + 2x^2 + 4x$$

The solution can be obtained by using both the scalar and additive properties from 9.

$$\begin{aligned} f_1(x) &= x^2 + 2x \\ f_2(x) &= 6e^{2x} \\ \ddot{y} + \dot{y} &= k_1 f_1(x) + k_2 f_2(x) \,\Big|\, k_1 = 2,\, k_2 = -\frac{1}{6} \\ \implies k_1 y_1 + k_2 y_2 &= \frac{2}{3}x^3 - \frac{1}{6}e^{2x} \end{aligned}$$

10 October 24: More Non-homogenous Equation

We will have a quiz this Wednesday. Our midterm average was 20.6 out of 25. The final exam will be 2 hours on BlackBoard using Respondus.

In the previous week, we learned sections 3.3 to 3.6: reduction of order and homogenous equation solutions.

If we know any two solution y_1 and y_2, then

$$y = c_1 y_1 + c_2 y_2$$

if they are linearly independent. The above equation can be used to illustrate the original differential equation. If y_1 is known, then by the reduction of order method 8.0.2 we can find

$$y_2(x) = \nu(x) \cdot y_1(x)$$

If the both solutions are unknown, then we can use 8. If the determinant is negative, then we can use Euler's formula to convert to real numbers as shown in 9.0.1.

For non-homogenous equations, we can use the input-output systems where we can use the additive and scalar properties as shown in the principles of superposition for non-homogenous equations 9.

Example 10.1. *Given the equations*

$$\begin{aligned} \ddot{y} + y &= 17e^{4x} &&= f_1(x) \\ \ddot{y} + y &= x^3 + 6x &&= f_2(x) \end{aligned}$$

have the solutions $y_1(x) = e^{4x}$ *and* $y_2(x) = x^3$*, then:*

$$\begin{aligned} \ddot{y} + y &= k_1 y_1(x) + k_2 y_2(x) \\ &= 17e^{4x} + 5x^3 \end{aligned}$$

Today's lecture will be about non-homogenous equations. Similar to the existence and uniqueness of homogenous equations, shown in 6:

Solution of Non-Homogenous Equations

If $y_p(x)$ is a particular solution of

$$\ddot{y} + p(x)\dot{y} + q(x)y = f(x)$$

and $y_h(x)$ is a general solution of

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0$$

then any solution of $Y(x)$ of 9.0.3 can be expressed as

$$Y(x) = y_h(x) + y_p(x) = (c_1 y_1(x) + c_2 y_2(x)) + y_p(x)$$

where y_1 and y_2 are linearly independent solutions of 8.0.1.

Note that the above does not guarantee uniqueness of solutions, thought it guarantees the existence of a solution. If Y is given, though, then it can be unique.

Example 10.2. *Given* $\ddot{y} - 2\dot{y} - 8y = 9x \cdot e^x$ *and that a solution* $y_p(x) = -x \cdot e^x$ *exists, then:*

$$\begin{aligned}
y(x) &= y_h(x) + y_p(x) \\
y_h(x) : \ddot{y} - 2\dot{y} - 8y &= 0 \\
m &= 4, -2 \\
\implies y_h(x) &= c_1 e^{-2x} + c_2 e^{4x} \\
\implies y(x) &= c_1 e^{-2x} + c_2 e^{4x} - xe^x
\end{aligned}$$

Notice that, in the above case, there are infinitely many arbitrary solutions due to the combinations of c_1 and c_2. However, if we placed an initial condition:

Example 10.3. *Going from the previous example, considering* $y(0) = 1$ *and* $\dot{y}(0) = 0$*, then:*

$$\begin{aligned}
y(x) = 0 : c_1 + c_2 &= 0 \\
\dot{y}(x) = 0 : -2c_1 e^{-2x} + 4c_2 e^{4x} - e^x - xe^x &= 0 \\
-2c_1 + 4c_2 - 1 &= 0
\end{aligned}$$

yields c_1 *and* c_2 *are both* $\frac{1}{2}$.

If we are given a non-homogenous equation with $f(x)$ *as a constant, then we can find* $y_p(x)$ *to be:*

$$y_p(x) = f(x) \tag{10.0.1}$$

if no other coefficients are present.

11 October 26: Method of Undetermined Parameters

This is section 3.7, which solves a particular solution of $y_p(x)$ via the method of undetermined coefficient (there is another part called the method of variation of parameters). A non-homogenous equation has a homogenous part 1.3.6, and a function $f(x)$. We can add any $y_h(x)$ with $y_p(x)$ to obtain the solution $y(x)$ because:

$$y = y_h + y_p \tag{11.0.1}$$

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0 \tag{11.0.2}$$

$$\Rightarrow (y_h + y_p)'' + p(x)(y_h + y_p)' + q(x)(y_h + y_p) = 0 \tag{11.0.3}$$

$$\Rightarrow (\ddot{y}_h + p(x)\dot{y}_h + q(x)y_h) + (\ddot{y}_p + p(x)\dot{y}_p + q(x)y_p) = 0 \tag{11.0.4}$$

$$\Rightarrow 0 + f(x) = 0 \tag{11.0.5}$$

This method works for all coefficients that are, or are the combinations of:

$$x^n,\ e^{ax},\ \sin ax,\ \cos ax \tag{11.0.6}$$

The $y_p(x)$ candidates for a given $f(x)$ is given in table 1. Once we identify the equivalent, we have to:

1. Replace the homogenous side of the equation (left side) with the $y_p(x)$ candidate (this includes derivative computation).
2. Solve the left side so that it matches the right side (non-homogenous side).

Note that the capital constants are the constants that we are trying to determine, while the small letters are what we are given. Additionally, the S in x^S is the number of common solutions between the $f(x)$ ($y_p(x)$) and $y_h(x)$.

$f(x)$	$y_p(x)$ candidate
a	$x^S \cdot A$ (constant)
$a_0x^n + a_1x^{n-1} + \cdots a_n$	$x^S \cdot (A_0x^n + A_1x^{n-1} + \cdots A_n)$
$ae^{\alpha x}$	$x^S \cdot Ae^{\alpha x}$
$a\cos kx + b\sin kx$	$x^S \cdot A\cos kx + B\sin kx$

Table 1: Method of Undetermined Parameters y_p Candidates

Example 11.1. *Determine the $y_p(x)$ for $\ddot{y} - 2\dot{y} - 3y = 4e^{-x}$. Step 1 is to find the $y_p(x)$ candidate:*

$$y_h = x^S \cdot Ae^{-x}$$

where $S = 1$ because the homogenous part shares the component e^{-x}:

$$y_h(x) = c_1e^{-x} + c_2e^{3x}$$

Notice that e^{-x} in $y_p(x)$ appears once in $y_h(x)$, therefore $S = 1$. If, instead, $f(x) = 4e^{-x}+5e^{3x}$, then we have to use the principle of super position to break f down into f_1 and f_2. Meanwhile, the solution to the original problem statement is:

$$\begin{aligned}
\ddot{y} - 2\dot{y} - 3y &= 4e^{-x} \\
y_h &= x \cdot Ae^{-x} \\
\dot{y}_h &= A(-x+1)e^{-x} \text{ product rule} \\
\ddot{y}_h &= A\left[-1 \cdot e^{-x} + (-x+1)(-e^{-x})\right] \text{ product rule} \\
&= A(x-2)e^{-x}
\end{aligned}$$

When we plug this into the left-hand side of the equation, we obtain:

$$\begin{aligned}
A(x-2)e^{-x} - 2A(-x+1)e^{-x} - 3Axe^{-x} &= 4e^{-x} \\
\Rightarrow Ae^{-x}\left[(x-2) - 2(-x+1) - 3x\right] &= 4e^{-x} \\
\Rightarrow -4Ae^{-x} &= 4e^{-x} \\
\therefore A &= \boxed{-1}
\end{aligned}$$

Another example:

Example 11.2. *We are given* $\ddot{y} + 2\dot{y} + y = 4e^{-x}$*, so first find the solutions of the homogenous part:*

$$\begin{aligned}
\ddot{y} + 2\dot{y} + y &= 4e^{-x} \\
m^2 + 2m + 1 &= 0 \\
\Rightarrow m &= -1 \\
y_h(x) &= c_1 e^{-x} + c_2 x e^{-x}
\end{aligned}$$

Using that, our candidate becomes

$$y_p(x) = x^2 \cdot Ae^{-x}$$

Solving, we will get $A = 2$

An example with trigonometric functions:

Example 11.3.

$$\begin{aligned}
\ddot{y} + \dot{y} &= 3\sin 5x \\
m^2 + 1 &= 0 \\
m &= \pm i \\
y_h(x) &= c_1 \cos x + c_2 \sin x \\
\therefore y_p(x) &= A\cos 5x + B\sin 5x \\
\dot{y}_p &= -5A\sin 5x + 5B\cos 5x \\
\ddot{y}_p &= -25A\cos 5x - 25B\sin 5x
\end{aligned}$$

Which yields

$$\begin{aligned}
(-25A\cos 5x - 25B\sin 5x) + (-5A\sin 5x + 5B\cos 5x) &= 3\sin 5x \\
\Rightarrow (-25B - 5A)\sin 5x + (-25A + 5B)\cos 5x &= 3\sin 5x
\end{aligned}$$

We now have to solve the simultaneous equation:

$$\begin{aligned}
-25A + 5B &= 0 \\
-25B - 5A &= 3 \\
\Rightarrow -25A + 25A + 5B + 125B &= -15 \\
\Rightarrow B &= -\frac{15}{130} \\
&= \boxed{-\frac{3}{26}} \\
25A &= 5 \cdot -\frac{3}{26} \\
5A &= -\frac{3}{26} \\
A &= \boxed{-\frac{3}{5 \cdot 26}}
\end{aligned}$$

12 October 31: Method of Variations

The general form of non-homogenous solution, for us, will come as:

$$
\begin{aligned}
y_h(x) &= c_1 e^{r_1 x} + c_2 e^{r_2 x} && (12.0.1)\\
\text{or } &= c_1 e^{rx} + c_2 x e^{rx} && (12.0.2)\\
\text{or } &= c_1 e^{px}\cos qx + c_2 e^{px}\cos qx && (12.0.3)
\end{aligned}
$$

The method of undetermined coefficients will only work for coefficients only work if the non-homogenous function is:

$$f(x) = x^n,\ \sin ax,\ \cos bx,\ e^{mx} \tag{12.0.4}$$

The method of variation of parameters, in contrast, can work. An example (still) of method of undetermined coefficients:

Example 12.1.

$$
\begin{aligned}
\ddot{y} + y &= 3\sin 5x\\
m^2 + 1 &= 0\\
m &= \pm i\\
y_h(x) &= c_1\cos x + c_2\sin x\\
\therefore y_p(x) &= A\cos 5x + B\sin 5x\\
\dot{y}_p &= -5A\sin 5x + 5B\cos 5x\\
\ddot{y}_p &= -25A\cos 5x - 25B\sin 5x
\end{aligned}
$$

Which yields

$$
\begin{aligned}
\ddot{y}_p + y_p &= 3\sin 5x\\
-24A\cos 5x - 24B\sin 5x &= 3\sin 5x + 0\cdot\cos 5x\\
\therefore B &= \boxed{0},\\
A &= \boxed{-\frac{1}{8}}
\end{aligned}
$$

Another example, but this has shared solution:

Example 12.2.

$$
\begin{aligned}
\ddot{y} + y &= 3\sin x\\
y(x) &= y_h(x) + y_p(x)\\
y_h(x):\ \ddot{y} + y &= 0\\
m^2 + 1 &= 0\\
m &= \pm i\\
y_h(x) &= c_1\cos x + c_2\sin x
\end{aligned}
$$

Notice that the $\cos x$ *is present here, and in*

$$A\cos x + B\sin x$$

therefore we multiply the whole thing with x^1*:*

$$\begin{aligned} y_p(x) &= x(A\cos x + B\sin x) \\ 3\sin x &= x(A\cos x + B\sin x) \\ \therefore y_p(x) &= -\frac{3}{2}x\cos x \end{aligned}$$

In the method of variation of parameters, we need to know the solutions of the homogenous set, given as

$$y_h = c_1y_1 + c_2y_2 \tag{12.0.5}$$

then the homogenous part becomes

$$y_p = v_1y_1 + v_2y_2$$

Plugging in y_p, we get

$$\begin{aligned} \ddot{y}_p + p\dot{y}_p + qy_p &= f(x) \\ \dot{y}_p &= v_1'y_1 + v_1\dot{y}_1 + v_2'y_2 + v_2y_2' \\ &= (v_1'y_1 + v_2'y_2) + (v_1y_1' + v_2y_2') \end{aligned}$$

We *assume* that

$$v_1'y_1 + v_2'y_2 = 0$$

in order to not have to compute the second derivative. Therefore, we get $\dot{y}_p = v_1y_1' + v_2y_2'$, and $\ddot{y}_p = (v_1y_1'' + v_2y_2'') + (v_1'y_1' + v_2'y_2')$.

$$
\begin{aligned}
1 : & v_1' y_1 + v_2' y_2 && = 0 && (12.0.6)\\
2 : & v_1' y_1' + v_2' y_2' && = 0 && (12.0.7)\\
& (1) \times y_2' + (2) \times y_2 && && (12.0.8)
\end{aligned}
$$

The v are given by the Wronskian:

$$
v_2' = \frac{y_1 f}{W(y_1, y_2)} \tag{12.0.9}
$$

$$
v_1' = -\frac{y_2 f}{W(y_1 y_2)} \tag{12.0.10}
$$

An example:

Example 12.3. *Given the following:*

$$\ddot{y} - 2\dot{y} - 3y = 2x^2$$
$$y_h(x) = c_1 e^{-x} + c_2 e^{3x}$$
$$y_p(x) = -\frac{2}{3}^2 + \frac{8}{9}x - \frac{28}{27}$$

Re-find y_p, but using method of variation of parameters: We know that

$$y_p(x) = v_1 y_1 + v_2 y_2$$

we can compute the Wronskian:

$$\begin{aligned} W(y_1, y_2) &= \begin{vmatrix} e^{-x} & e^{3x} \\ -e^{-x} & 3e^{3x} \end{vmatrix} \\ &= 3e^{2x} - (-e^{2x}) \\ &= 4e^{2x} \end{aligned}$$

And using the Wronskian, compute the v':

$$\begin{aligned} v_1' &= -\frac{y_2 f}{W(y_1 y_2)} \\ &= -\frac{e^{3x} 2x^2}{4e^{2x}} \\ &= -\frac{1}{2}x^2 e^x \\ v_2' &= \frac{y_1 f}{W(y_1, y_2)} \\ &= \frac{e^{-x} 2x^2}{4e^{2x}} \\ &= \frac{1}{2}x^2 e^{-3x} \end{aligned}$$

Next, we will have to integrate using integration by parts. We start with v_1:

$$\begin{aligned} v_1 &= -\int \left(\frac{1}{2}x^2 e^x\right) dx \\ f &= \frac{1}{2}x^2 \\ \dot{g} &= e^x \\ v_1 &= -\left(\frac{1}{2}x^2 e^x - \int x e^x \, dx\right) \\ &= -\frac{1}{2}x^2 e^x + \left(x e^x - \int e^x \, dx\right) \\ &= -\frac{1}{2}x^2 e^x + x e^x - e^x \\ &= e^x \left(-\frac{1}{2}x^2 + x - 1\right) \end{aligned}$$

Now, for v_2:

$$
\begin{aligned}
v_2 &= \int \frac{1}{2}x^2 e^{-3x}\,dx \\
f &= \frac{1}{2}x^2 \\
\dot{g} &= e^{-3x} \\
v_2 &= \frac{1}{2}x^2\frac{e^{-3x}}{-3} - \int x\frac{e^{-3x}}{3}\,dx \\
&= -\frac{x^2e^{-3x}}{6} + \frac{1}{3}\int xe^{-3x}\,dx \\
&= -\frac{x^2e^{-3x}}{6} + \frac{1}{3}\left[x\frac{e^{-3x}}{-3} - \int \frac{e^{-3x}}{-3}\,dx\right] \\
&= -\frac{x^2e^{-3x}}{6} - \frac{1}{9}xe^{-3x} + \frac{1}{9}\int e^{-3x}\,dx \\
&= -\frac{1}{6}x^2e^{-3x} - \frac{1}{9}xe^{-3x} - \frac{1}{27}e^{-3x} \\
&= e^{-3x}\left[-\frac{1}{6}x^2 - \frac{1}{9}x - \frac{1}{27}\right]
\end{aligned}
$$

And finally, combining all these into the original equation for y_p, we get:

$$
\begin{aligned}
y_p &= v_1y_1 + v_2y_2 \\
&= e^x\left(-\frac{1}{2}x^2 + x - 1\right) \times e^{-x} + e^{-3x}\left[-\frac{1}{6}x^2 - \frac{1}{9}x - \frac{1}{27}\right] \times e^{3x} \\
&= \boxed{-\frac{2}{3}^2 + \frac{8}{9}x - \frac{28}{27}}
\end{aligned}
$$

Exercise that we are given to try on our own:

Example 12.4. *Compute the solutions for $\ddot{y} - 2\dot{y} + 2y = e^x \cdot \sin x$. First, we compute the homogenous part:*

$$
\begin{aligned}
m^2 - 2m + 2 &= 0 \\
m &= 1 \pm i \\
y_h(x) &= c_1e^x\cos x + c_2e^x\sin x
\end{aligned}
$$

A preliminary y_p according to 1:

$$\hat{y}_p(x) = (Ae^x\cos x + Be^x\sin x)$$

But notice how the $(e^x\cos x + e^x\sin x)$ component is present in the $y_h(x)$ section. (We are comparing the whole thing) Therefore we multiply the whole thing with x:

$$\hat{y}_p(x) = x \cdot (Ae^x\cos x + Be^x\sin x)$$

13 November 2: Method of Varying Parameters and Simple Harmonic Motion

Example problem (3.7.15) that I went over during office hour:

$$\sin x(-2A_1 - 2(B_1x + B_2)) + \cos x(2B_2 - 2(A_1x + A_2)) = x\sin x$$
$$\sin x(-2B_1x + (-2A_1 - 2B_2)) + \cos x(-2A_1x + (2B_2 - 2A_2)) = x\sin x$$
$$-2B_1 = 1$$
$$-2A_1 - 2B_2 = 0$$
$$-2A_1 = 0$$
$$2B_2 - 2A_2 = 0$$

Therefore, substituting:

$$B_1 = -\frac{1}{2}$$
$$A_1 = 0$$
$$B_2 = 0$$
$$A_2 = 0$$

Method of Variation of Parameters

For the differential equation

$$\ddot{y} + p(x)\dot{y} + q(x)y = f(x)$$

the method of variation of parameter tells us that, if y_1 and y_2 are the solutions of the homogenous part y_h, then

$$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{13.0.1}$$

where we can determine v using the two equations:

$$v_1'y_1 + v_2'y_2 = 0 \tag{13.0.2}$$
$$v_1'y_1' + v_2'y_2' = f(x) \tag{13.0.3}$$

where

$$v_1' = -\frac{y_2 f}{W(y_1, y_2)} \tag{13.0.4}$$
$$v_2' = \frac{y_1 f}{W(y_1, y_2)} \tag{13.0.5}$$

Example 13.1. *Given the homogenous solution $y_h = c_1e^x\cos x + c_2e^x\sin x$ for the problem $\ddot{y} - 2\dot{y} + 2y = e^x\sin x$. To solve this using the method of variation* *of parameters, we compute the Wronskian of*

$$y_1 = e^x \cos x$$
$$y_2 = e^x \sin x$$

to get

$$\begin{aligned}
W(y_1, y_2) &= \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix} \\
&= e^{2x}(\cancel{\cos x \sin x} + \cos^2 x) - e^{2x}(\cancel{\cos x \sin x} - \sin^2 x) \\
&= e^{2x}(\cos^2 x + \sin^2 x) \\
&= e^{2x}
\end{aligned}$$

Using the Wronskian, we obtain the v':

$$\begin{aligned}
v_2' &= \frac{e^x \cos x e^x \sin x}{e^{2x}} \\
&= \sin x \cos x \\
&= \frac{1}{2} \sin 2x \\
v_1' &= -\frac{e^x \sin x e^x \sin x}{e^{2x}} \\
&= -\sin^2 x \\
&= \frac{1}{2}(\cos 2x - 1)
\end{aligned}$$

And then we can compute their anti-derivative:

$$\begin{aligned}
v_2 &= \int \frac{1}{2} \sin 2x \, dx \\
&= \frac{1}{2} \times -\frac{\cos 2x}{2} \\
&= -\frac{1}{4} \cos 2x \\
v_1 &= \int \frac{1}{2}(\cos 2x - 1) \, dx \\
&= -\frac{1}{2}x + \frac{\sin 2x}{4}
\end{aligned}$$

Plugging these back in to the equation 13:

$$\begin{aligned}
y_p(x) &= v_1(x)y_1(x) + v_2(x)y_2(x) \\
&= \left(-\frac{1}{4}\cos 2x\right)(e^x \cos x) + \left(-\frac{1}{2}x + \frac{\sin 2x}{4}\right)(e^x \sin x) \\
&= \left(-\frac{1}{4}(1 - 2\sin^2 x)\right)(e^x \cos x) + \left(-\frac{1}{2}x + \frac{\sin x \cdot \cos x}{4}\right)(e^x \sin x) \\
&= -\frac{1}{2}xe^x \cos x + \frac{1}{2}e^x \sin x \cos^2 x - \frac{1}{4}e^x \sin x + \frac{1}{2}e^x \sin^3 x \\
&= -\frac{1}{2}xe^x \cos x + \frac{1}{4}e^x \sin x
\end{aligned}$$

Note that we already see $e^x \sin x$ in one of the homogenous solutions, y_2, that is, it is a repeat, and we have to ignore repeats. Therefore, the full solution is:

$$y = c_1 e^x \cos x + c_2 e^x \sin x - \frac{1}{2}xe^x \cos x$$

Fun fact, the derivative of the product of three functions is:

$$(fgh)' = f'gh + fg'h + fgh' \tag{13.0.6}$$

The equilibrium position is the position where, if that is the initial state, there is no motion. The position can be both positive and negative. Generally, down is considered to be positive. Velocity and acceleration are given as

$$\dot{y} = \frac{dy}{dt} \tag{13.0.7}$$

$$\ddot{y} = \frac{d^2y}{dt^2} \tag{13.0.8}$$

Note that m is the mass and k is the constant of proportionality for the restoring force. The maximum force is directly proportional to the displacement y. Therefore, we obtain:

$$m \cdot \ddot{y} + k \cdot u = 0 \tag{13.0.9}$$

As an example:

Example 13.2. *For the parameters $m = 1$, $k = 16$:*

$$\begin{aligned}
\ddot{y} + 16y &= 0 \\
m^2 + 16 &= 0 \\
m &= 0 \pm \sqrt{-16} \\
&= 0 \pm 4i \\
y &= c_1 \cos 4t + c_2 \sin 4t
\end{aligned}$$

With the initial conditions $y_0 = 1$ and $\dot{y}_0 = -4$:

$$\begin{aligned} c_1 \cos 4t + c_2 \sin 4t &= 1 \\ -4c_1 \sin 4t + 4c_2 \cos 4t &= -4 \\ \Rightarrow c_1 \sin 4t - c_2 \cos 4t &= 1 \\ \therefore c_1 \cos 4t + c_2 \sin 4t &= c_1 \sin 4t - c_2 \cos 4t \\ \Rightarrow (c_1 + c_2) \cos 4t &= (c_1 - c_2) \sin 4t \end{aligned}$$

The circular frequency is the number of oscillations in time 2π (w_o), while the natural frequency is the number of oscillations in unit time

$$f = \frac{w_o}{2\pi} \tag{13.0.10}$$

The period is the reciprocal of the natural frequency $T = \frac{1}{f}$.

14 November 7: Further Harmonic Oscillator

This week's quiz will be on 3.8 (Variation of Parameters) and 3.9 (Harmonic Oscillator).

Example 14.1. *Given $\ddot{y} + 16y = 0$, with initial conditions $y(0) = 1$ and $\dot{y}(0) = -4$:*

$$\begin{aligned} y &= c_1 \cos t + c_2 \sin t \\ \cos(\alpha - \beta) &= \cos\alpha \cdot \cos\beta + \sin\alpha \cdot \sin\beta \\ \Rightarrow \cos(t - \delta) &= \cos t \cdot \cos\delta + \sin t \cdot \sin\delta \\ \cos^2\delta + \sin^2\delta &= 1 \\ \therefore c_1 \cos t + c_2 \sin t &= \sqrt{2}\left(\frac{\cos t}{\sqrt{2}} - \frac{\sin t}{\sqrt{2}}\right) \\ \textit{where } \cos\delta &= \frac{1}{\sqrt{2}}, \\ \sin\delta &= -\frac{1}{\sqrt{2}} \textit{how is this negative?} \end{aligned}$$

Much confused :)

Note that

$$\omega_o = \sqrt{\frac{k}{m}} \tag{14.0.1}$$

The phase angle is given as

$$\delta = \tan^{-1}\left(\frac{c_2}{c_1}\right) \tag{14.0.2}$$

And the general solution is:

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega_o t \tag{14.0.3}$$

Slug is a useless unit. It's value is:

$$1 \text{ slug} = \frac{1 \text{ lb}}{1 \text{ ft/s}^2} = \frac{32 \text{ lb}}{32 \text{ ft/s}^2} \tag{14.0.4}$$

The extended system for damped/driven oscillations of hanging spring is:

$$m\ddot{y} = F_g + F_s + F_d + F_e \tag{14.0.5}$$
$$= mg - k(\Delta L + y) - d\dot{y} + F(t) \tag{14.0.6}$$

where F_g is the force of gravity on the spring, F_s is the spring force, F_d is the dampening force, and $F_e = F(t)$ is the driving force. Note that y is taken from the equilibrium position. d is the constant for dampening, and ΔL is distance of equilibrium to starting position (no mass). Hence, if we consider

$$mg = k\Delta L \text{ at equilibrium} \tag{14.0.7}$$

we can rewrite the equations as

$$m\ddot{y} + d\dot{y} + ky = F(t) \tag{14.0.8}$$

Dampening states:

$$d^2 - 4mk < 0 \qquad \text{is under damped} \tag{14.0.9}$$
$$d^2 - 4mk = 0 \qquad \text{is critically damped} \tag{14.0.10}$$
$$d^2 - 4mk > 0 \qquad \text{is over damped} \tag{14.0.11}$$

For the latter two, the spring-mass system will eventually come to a halt.

15 November 9: Damping and Laplace Transform

Vertical Spring Mass System Formulae

A vertical spring mass system written in the form of

$$m\ddot{y} = F_g + F_s + F_d + F_e \tag{15.0.1}$$
$$= mg - k(\Delta L + y) - d\dot{y} + F(t) \tag{15.0.2}$$

can be solved as

$$y(t) = c_1 \cos \omega t + c_2 \sin \omega_o t \tag{15.0.3}$$

which can be re-written as

$$y(t) = R \cos(\omega t - \delta) \tag{15.0.4}$$

where

$$R = \sqrt{c_1^2 + c_2^2} \tag{15.0.5}$$
$$\omega = \sqrt{\frac{k}{m}} \tag{15.0.6}$$
$$\delta = \tan^{-1}\left(\frac{c_1}{c_2}\right) \tag{15.0.7}$$

In an under damped scenario 14.0.9, oscillations continue, but slows down eventually to zero.

In a critically damped scenario 14.0.10, our equation for the position 15.0.3 changes to:

$$y(t) = c_1 e^{-t} + c_2 t e^{-t} \tag{15.0.8}$$

and the practical result is a lack of oscillation while reaching zero (does not cross the x-axis)

In an over damped case, the corresponding homogenous situation becomes:

$$y_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{15.0.9}$$

where

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{15.0.10}$$

which results in no oscillation, but also reaching zero takes a long time.

Laplace Transform can be used to alternatively find the solution to a differential equation. Laplace transform is generally denoted with curly brackets.

Laplace Transform

For a function $f(t)$ on $(0, \infty)$, a Laplace transform of $f(t)$ is denoted as and equal to:

$$\mathcal{L}\{f\} = F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt \tag{15.0.11}$$

for $s > 0$. Note that s is a constant.

Example 15.1. *Laplace transform of* 1 *is:*

$$\mathcal{L}\{1\} = F(s) = \int_0^\infty e^{-st} dt = \left[-\frac{1}{s} e^{-st} \right]_{t=0}^{t\to\infty} = \frac{1}{s}$$

Another example, using integration of parts via tableau method:

Example 15.2.

$$\mathcal{L}\{t\} = F(s) = \int_0^\infty te^{-st}\, dt$$

u	dv
t	e^{-st}
-1	$-\frac{1}{s}e^{-st}$
0	$\frac{1}{s^2}e^{-st}$

Table 2: Tableau Method for Integration of Parts

$$= \left[\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_{t=0}^{t\to\infty} = \frac{1}{s^2}$$

Example 15.3.

$$\begin{aligned} \mathcal{L}\left\{e^{at}\right\} &= F(s) \\ &= \int_0^\infty e^{at} \cdot e^{-st}\, dt \\ &= \int_0^\infty e^{t(a-s)}\, dt \\ &= \left[\frac{e^{t(a-s)}}{a-s} \right]_{t=0}^{t\to\infty} \\ &= \frac{1}{s-a} \end{aligned}$$

16 November 14: Laplace Transform

The variable s can have a restriction. One property of the Laplace transform is:

$$\mathcal{L}\{f'\} = s \cdot F(s) - f(0) \ni F(s) = \mathcal{L}\{f\} \tag{16.0.1}$$

The quiz will be on 3.10 and 5.1, in groups.

Laplace transform is useful in cases where there is discontinuity in the original function.

Example 16.1. *Given the step function:*

$$f(t) = \begin{cases} 1 & t \geq 5 \\ 0 & 0 \leq t < 5 \end{cases} \tag{16.0.2}$$

In this case, we can assume that f is a unit step function denoted as:

$$f(t) = u(t-5) \tag{16.0.3}$$

and hence be denoted with the Laplace function:

$$\begin{aligned}
\mathcal{L}\{f\} &= \mathcal{L}\{u(t-5)\} \\
&= \int_0^\infty f(t)e^{-st}\,dt \\
&= \int_0^5 0e^{-st}\,dt + \int_5^\infty 1e^{-st}\,dt \\
&= \int_5^\infty e^{-st}\,dt \\
&= \left[-\frac{1}{s}e^{-st}\right]_{t=5}^{t\to\infty} \\
&= [0] - \left[-\frac{1}{s}e^{-5s}\right] \\
&= \frac{e^{-5s}}{s}
\end{aligned}$$

Therefore:

$$\begin{aligned}
\mathcal{L}\{u(t-5)\} &= \frac{e^{-5s}}{s} \\
\mathcal{L}\{u(t-c)\} &= \frac{e^{-cs}}{s}
\end{aligned}$$

$f(t)$	$F(s)$	domain of $F(s)$
1	$\frac{1}{s}$	$s > 0$
$t^n \ni n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$	$s > 0$
$t^p \ni p > -1$	$\frac{p(p+1)}{s^{p+1}}$	$s > 0$
e^{at}	$\frac{1}{s-a}$	$s > a$
$e^{at} \cdot t^n \ni n \in \mathbb{N}$	$\frac{n!}{(s-a)^{n+1}}$	$s > a$
$\sin bt$	$\frac{b}{s^2+b^2}$	$s > 0$
$\cos bt$	$\frac{s}{s^2+b^2}$	$s > 0$
$\sinh bt$	$\frac{b}{s^2-b^2}$	$s > \|b\|$
$\cosh bt$	$\frac{s}{s^2-b^2}$	$s > \|b\|$
$e^{at} \cdot \sin bt$	$\frac{b}{(s-a)^2+b^2}$	$s > a$
$e^{at} \cdot \cos bt$	$\frac{(s-a)}{(s-a)^2+b^2}$	$s > a$
$u(t-c)$	$\frac{e^{-cs}}{s}$	$s > 0$
$u(t-c) \cdot f(t-c)$	$e^{-cs} \cdot F(s)$	
$\int_0^t f(t-\tau) \cdot g(\tau) d\tau$	$F(s)G(s)$	
$\delta(t-c)$	e^{-cs}	
$\frac{d^n}{dt^n} f(t)$	$s^n \cdot F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots f^{(n-1)}(0)$	
$t^n \cdot f(t)$	$(-1)^n \cdot F^n(s)$	

Example 16.2. *Express the following function:*

$$f(t) = \begin{cases} -2t + 39 & 0 \leq t \leq 10 \\ 2t - 1 & t \geq 10 \end{cases}$$

in terms of u using Laplace transform.

This can be re-written with 16 as:

$$u(t-10) = \begin{cases} 0 & 0 \leq t \leq 10 \\ 1 & t \geq 10 \end{cases}$$

and then later transformed into

$$1 - u(t-10) = \begin{cases} 1 & 0 \leq t \leq 10 \\ 0 & t \geq 10 \end{cases}$$

The function f is a composition of f_1 and f_2 where

$$f_1(t) = \begin{cases} -2t + 39 & 0 \leq t \leq 10 \\ 0 & t < 10 \end{cases}$$

$$f_2(t) = \begin{cases} 0 & 0 \leq t < 10 \\ 2t - 1 & t \geq 10 \end{cases}$$

Now, we can combine the expression that has t with the unit function:

$$f(t) = (-2t + 39) \cdot (1 - u(t - 10)) + (2t - 1) \cdot u(t - 10)$$

Their Laplace Transform would be

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{(-2t + 39) \cdot (1 - u(t - 10))\} + \mathcal{L}\{(2t - 1) \cdot u(t - 10)\} \\ &= \mathcal{L}\{2t - 39\} \cdot \end{aligned}$$

Unit Step Function $u(t)$

The function $u(t)$ is the unit step function that is defined as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \tag{16.0.4}$$

17 November 16: Properties of Laplace Transform

Example 17.1. *Laplace transform of:*

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & \pi \leq t \end{cases}$$

using

$$\mathcal{L}\{u(t - c) \cdot f(t - c)\} = e^{-cs} F(s)$$

Our main equation can be written in unit piecewise function as

$$f(t) = \sin t \cdot (1 - u(t - \pi))$$

From there, we can compute the Laplace transform:

$$\begin{aligned}
\mathcal{L}\{f(t)\} &= \mathcal{L}\{\sin t - u(t-\pi)\sin t\} \\
&= \mathcal{L}\{\sin t\} - \mathcal{L}\{u(t-\pi)\sin t\} \\
&= \frac{1}{s^2+1} - e^{-\pi s}\cdot \mathcal{L}\{\sin t\} \\
f(t-\pi) &= \sin t \\
\therefore f(t) &= \sin(t+\pi) \\
&= -\sin t \\
\mathcal{L}\{f(t)\} &= \frac{1}{s^2+1} + e^{-\pi s}\cdot \frac{1}{s^2+1} \\
&= \frac{1}{s^2+1}(1+e^{-\pi s}) \ni s > 0
\end{aligned}$$

Derivative property:

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0) \tag{17.0.1}$$

$$\begin{aligned}
\implies \mathcal{L}\{f^{(n)}\} &= s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \\
&\quad s^{n-3}f''(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)
\end{aligned} \tag{17.0.2}$$

Translation property:

$$\mathcal{L}\{e^{at}\cdot f(t)\} = F(s-a) \tag{17.0.3}$$

$$F(s) = \mathcal{L}\{f(t)\} \tag{17.0.4}$$

Assuming $F(s)$ on $s > \beta$,

$$\mathcal{L}\{e^{at}\cdot f(t)\} = \int_0^\infty e^{at}\cdot f(t)e^{-st}\,dt \tag{17.0.5}$$

$$= \int_0^8 f(t)\cdot e^{t(a-s)}\,dt \tag{17.0.6}$$

$$= F(s-a) \ni s > \beta + b \tag{17.0.7}$$

Derivative of Transform:

$$\mathcal{L}\{t^n\cdot f(t)\} = (-1)^n\frac{d^n}{ds^n}F(s) \tag{17.0.8}$$

If $f(t) \sim e^{\beta t}$ for $s > \beta$

$$\frac{d}{ds}F(s) = \frac{d}{ds}\int_0^\infty e^{-st}\cdot f(t)\,dt \tag{17.0.9}$$

$$= \int_0^\infty (-t)e^{-st}\cdot f(t)\,dt \tag{17.0.10}$$

$$\frac{d}{ds^n}F(s) = (-1)^n\int_0^\infty t^n e^{-st}\cdot f(t)\,dt \tag{17.0.11}$$

List of Inverse Laplace Transforms

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 \tag{17.0.12}$$

$$\mathcal{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n \tag{17.0.13}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at} \tag{17.0.14}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2+b^2}\right\} = \sin bt \tag{17.0.15}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+b^2}\right\} = \cos bt \tag{17.0.16}$$

$$\mathcal{L}^{-1}\left\{\frac{b}{s^2-b^2}\right\} = \sinh bt \tag{17.0.17}$$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2-b^2}\right\} = \cosh bt \tag{17.0.18}$$

Example 17.2.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2+16}\right\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{4}{s^2+4^2}\right\} \\ &= \frac{1}{4}\sin 4t\end{aligned}$$

Example 17.3.

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+5}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+4}\right\} \\ &= e^{-t}\left(\frac{1}{2}\sin 2t\right)\end{aligned}$$

Example 17.4. *Using partial fraction decomposition:*

$$\begin{aligned}\frac{10}{s(s+5)} &= \frac{2}{s} - \frac{2}{s+5} \\ \Rightarrow \mathcal{L}^{-1}\left\{\frac{10}{s(s+5)}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} \\ &= 2 - 2e^{-5t}\end{aligned}$$

18 November 21: Initial Value Problems and Discontinuous Forcing Functions

Partial fraction decomposition for

$$\mathcal{L}^{-1}\left\{\frac{P(s)}{Q(s)}\right\} \tag{18.0.1}$$

can be solved by using:

1. If $Q(s) = as + b$, then
$$\frac{A}{as+b} \tag{18.0.2}$$

2. If $Q(s) = (as+b)^n$, then
$$\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \cdots + \frac{A_n}{(as+b)^n} \tag{18.0.3}$$

3. If $Q(s) = (as^2 + bs + c)$, then
$$\frac{As+B}{as^2+bs+c} \tag{18.0.4}$$

4. If $Q(s) = (as^2 + bs + c)^n$, then
$$\frac{A_1 s + B_1}{as^2+bs+c} + \frac{A_2 s + B_2}{(as^2+bs+c)^2} + \cdots + \frac{A_n s + B_n}{(as^2+bs+c)^n} \tag{18.0.5}$$

Example 18.1. *To solve*

$$\mathcal{L}^{-1}\left\{\frac{10}{s(s+5)}\right\}$$

we first do partial fraction decomposition

$$\begin{aligned}\frac{10}{s(s+5)} &= \frac{A}{s} + \frac{B}{s+5} \\ &= \frac{2}{s} - \frac{2}{s+5}\end{aligned}$$

to get:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{10}{s(s+5)}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s+5}\right\} \\ &= \boxed{2 - 2e^{-5t}}\end{aligned}$$

Example 18.2.

$$\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\}$$

$$\begin{aligned}\frac{8}{s(s^2+4)} &= \frac{A}{s} + \frac{Bs+C}{s^2+4} \\ &= \frac{2}{s} - \frac{2s}{s^2+4}\end{aligned}$$

Now doing the inverse Laplace transform:

$$\begin{aligned}\mathcal{L}^{-1}\left\{\frac{8}{s(s^2+4)}\right\} &= 2\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 2\mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} \\ &= \boxed{2 - 2\cos 2t}\end{aligned}$$

Example 18.3. *Do this at home:*

$$\mathcal{L}^{-1}\left\{\frac{s^2}{(s-9)^3}\right\}$$

Laplace transforms can be used to do initial value problems, that would have otherwise been solved by method of undetermined parameters or method of variations.

Example 18.4. *Solve the IVP:*

$$\begin{aligned}\ddot{y} + 9y &= 20e^{-t} \\ y(0) &= 0 \\ \dot{y}(0) &= 1 \\ \mathcal{L}\{\ddot{y}\} + 9\mathcal{L}\{y\} &= 20\mathcal{L}\left\{e^{-t}\right\} \\ \Rightarrow \left(s^2 \cdot Y(s) - s \cdot y(0) - \dot{y}(0)\right) + 9Y(s) &= \frac{20}{s+1} \\ \Rightarrow \left(s^2 Y(s) - 1\right) + 9Y(s) &= \frac{20}{s+1} \\ (s^2+9) \cdot Y(s) &= \frac{20}{s+1} + 1 \\ Y(s) &= \frac{20}{(s+1)(s^2+9)} + \frac{1}{s^2+9}\end{aligned}$$

Now do Inverse Laplace Transform:

$$\begin{aligned}\mathcal{L}^{-1}\{Y(s)\} &= \mathcal{L}^{-1}\left\{\frac{2}{s+1}\right\} + \mathcal{L}^{-1}\left\{\frac{-2s+2}{s^2+3^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+3^2}\right\} \\ \Rightarrow y &= 2e^{-t} - 2\cos 3t + \frac{2}{3}\sin 3t + \frac{1}{3}\sin 3t \\ &= \boxed{2e^{-t} - 2\cos 3t + \sin 3t}\end{aligned}$$

Example 18.5. *Solving:*

$$\begin{aligned}
\ddot{y} - 2\dot{y} - 3y &= 0\\
y(0) &= 0\\
\dot{y}(0) &= 1\\
\mathcal{L}\{\ddot{y}\} - 2\mathcal{L}\{\dot{y}\} - 3\mathcal{L}\{y\} &= 0\\
\left(s^2 \cdot Y(s) - s \cdot y(0) - \dot{y}(0)\right) - 2\left(s \cdot Y(s) - y(0)\right) - 3\left(Y(s)\right) &= 0\\
s^2 \cdot Y(s) - 2s \cdot Y(s) - 3Y(s) - 1 &= 0
\end{aligned}$$

We need to re-arrange and do partial fraction decomposition:

$$\begin{aligned}
Y(s) &= \frac{1}{s^2 - 2s - 3}\\
&= \frac{A}{s+1} + \frac{B}{s-3}\\
A + B &= 0\\
B - 3A &= 0\\
A &= -\frac{1}{4}\\
B &= \frac{1}{4}
\end{aligned}$$

Now, do inverse Laplace Transform:

$$\begin{aligned}
\mathcal{L}^{-1}\{Y(s)\} &= \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\}\\
&= \boxed{\frac{1}{4}e^{-t} - \frac{1}{4}e^{3t}}
\end{aligned}$$

Laplace transforms can be used to solve differential equations with discontinuity.

Example 18.6. *First converting to Laplace*

$$\begin{aligned}
\ddot{y} &= 1 - u(t-1)\\
y(0) &= 0\\
\dot{y}(0) &= 0\\
\mathcal{L}\{\ddot{y}\} &= \mathcal{L}\{1\} - \mathcal{L}\{u(t-1)\}\\
s^2 \cdot Y(s) - s \cdot \dot{y}(0) - y(0) &= \frac{1}{s} - \frac{e^{-s}}{s}\\
s^2 \cdot Y(s) &= \frac{1}{s} - \frac{e^{-s}}{s}\\
Y(s) &= \frac{1}{s^3} - \frac{e^{-s}}{s^3}
\end{aligned}$$

Then doing the inverse:

$$\begin{aligned}\mathcal{L}^{-1}\{Y(s)\} &= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} - \frac{1}{2}\mathcal{L}^{-1}\left\{e^{-s}\cdot\frac{2}{s^3}\right\} \\ &= \frac{1}{2}t^2 - \frac{1}{2}\cdot u(t-1)\cdot f(t-1) \ni f(t) = t^2 \\ &= \boxed{\frac{1}{2}t^2 - \frac{1}{2}u(t-1)(t-1)^2}\end{aligned}$$

19 November 23: Introduction to Linear Systems

We should read pages 323 to 328 to get a revision of linear algebra. We will later learn about converting an nth order differential equation into n first order differential equations. This produces a linear system of differential equations. An example of such system is:

$$\begin{aligned}\frac{dx_1}{dt} &= a_{11}(t)x_1 + a_{12}(t)x_2 + \cdots + a_{1n}(t)x_n + f_1(t) \\ \frac{dx_2}{dt} &= a_{21}(t)x_1 + a_{22}(t)x_2 + \cdots + a_{2n}(t)x_n + f_2(t) \\ \vdots &= \vdots \\ \frac{dx_n}{dt} &= a_{n1}(t)x_1 + a_{n2}(t)x_2 + \cdots + a_{nn}(t)x_n + f_n(t)\end{aligned}$$

In matrix form, the elements are denoted as $a_{ij}(t)$, where i is the row, and j is the column. we can simplify the above equation by calling a derivative on the left side:

$$\frac{d}{dt}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} = \begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}\end{bmatrix}\begin{bmatrix}x_1\\ \vdots \\ x_n\end{bmatrix} + \begin{bmatrix}f_1\\ \vdots \\ f_n\end{bmatrix} \tag{19.0.1}$$

Example 19.1. *Re-writing the following as a system of equations:*

$$\begin{aligned}\dot{x} &= 2y - 2 \\ \dot{y} &= 6x - 7y + 1\end{aligned}$$

Where x and y are both functions of t. This yields:

$$\begin{aligned}\frac{d}{dt}\begin{bmatrix}x\\ y\end{bmatrix} &= \begin{bmatrix}2y\\ 6x-7y\end{bmatrix} + \begin{bmatrix}-2\\ 1\end{bmatrix} \\ &= \begin{bmatrix}0 & 2\\ 6 & -7\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} + \begin{bmatrix}-2\\ 1\end{bmatrix}\end{aligned}$$

Example 19.2.

$$\begin{aligned}\dot{y} &= 3x - 4y\\ \dot{x} &= 5y - x + 10\\ \frac{d}{dt}\begin{bmatrix}x\\y\end{bmatrix} &= \begin{bmatrix}-1 & 5\\ 3 & -4\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \begin{bmatrix}10\\0\end{bmatrix}\end{aligned}$$

The D operator is used to represent the Leibniz notation for a derivative:

$$\begin{aligned}D &= \frac{d}{dt}\\ D^2 &= \frac{d^2}{dt^2}\\ \vdots &= \vdots\\ D^n &= \frac{d^n}{dt^n}\end{aligned}$$

Example 19.3. *Rewriting the following using the D operator:*

$$\begin{aligned}\ddot{y} - 2\dot{y} - 3y &= 0\\ D^2 \cdot y - 2D \cdot y - 3y &= 0\end{aligned}$$

which can be simplified as:

$$\begin{aligned}\left(D^2 - 2D - 3\right) y &= 0\\ (D+1)(D-3)y &= 0\end{aligned}$$

Example 19.4. *Unwrapping:*

$$\begin{aligned}D(D-2)\cdot\cos t &= (D^2 - 2D)\cos t\\ &= \frac{d^2}{dt^2}\cos t - 2\frac{d}{dt}\cos t\\ &= 2\sin t - \cos t\end{aligned}$$

An nth order equation can be converted to a system of first-order differential equations.

Example 19.5. *Converting to a first-order system of linear equations:*

$$\begin{aligned}\ddot{y} + y &= t\\ y(0) &= 1\\ \dot{y}(0) &= 0\end{aligned}$$

Letting $x_{n+1} = y^{(n)}$:

$$\begin{aligned}x_1 &= y\\ x_2 &= \dot{y}\\ \Rightarrow \dot{x}_1 &= \dot{y}\\ \Rightarrow \dot{x}_2 &= \ddot{y}\end{aligned}$$

Now we can write the linear system in matrix form:

$$\therefore \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix}$$

Example 19.6. *Higher power to lower:*

$$\begin{aligned} y^{(4)} - y = 0 & \\ x_i = y^{(i-1)} & \qquad \forall i \leq n \\ \dot{x}_i = y^{(i)} & \qquad \forall i \leq n \\ = x_{i-1} & \qquad \forall i \leq n-1 \end{aligned}$$

where n is the highest derivative. Therefore, our solution is:

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ x_4 \\ x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

Method of elimination can normally be used when we have around 2 to 3 variables. Suppose we have

$$\begin{aligned} Dx - y &= 0 \\ -x + Dy &= 0 \end{aligned}$$

We can treat it as a system of equations and use method of eliminations (multiplying the first equation with D)

$$\begin{aligned} D \cdot [\qquad & Dx - y = 0] \\ +[\qquad & -x + Dy = 0] \\ = \qquad & D^2x - x = 0 \\ = \qquad & \ddot{x} - x = 0 \end{aligned}$$

The zero matrix is denoted as $\mathbf{O}_{m\times n}$, which only has zeros. The identity matrix, $\mathbf{I}_{n\times n}$, which has ones on the main diagonal, and zeros elsewhere. It is equivalent to '1' in regular math:

$$\text{Regular:} \qquad a \cdot 1 = 1 \cdot a \qquad = a \tag{19.0.2}$$

$$\text{Matrices:} \qquad \mathbf{A} \cdot \mathbf{I} = \mathbf{I} \cdot \mathbf{A} \qquad = \mathbf{A} \tag{19.0.3}$$

For matrices, addition is element wise:

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \ni c_{ij} = a_{ij} + b_{ij} \tag{19.0.4}$$

and the same is true for subtraction. Note that small letters denote the element of a matrix, while big bold letters denote the matrix itself.

Matrix multiplication requires the number of columns n of the first to match the number of rows m of the second matrix. Matrix multiplication is the dot product of the individual rows of the first matrix, with the individual columns of the second matrix.

The determinant is denoted with straight bars. For a 2×2 matrix, it is:

$$|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \tag{19.0.5}$$

The inverse of a matrix is such that:

$$\mathbf{A}^{-1} \cdot \mathbf{A} = \mathbf{I} \tag{19.0.6}$$

analogous to

$$a \cdot a^{-1} = \frac{\not{a}}{\not{a}} = 1 \tag{19.0.7}$$

20 November 28: Basic Theory of First Order Linear Systems

The final exam is on the 12th of December, in class, and we are allowed to bring a formula sheet double-sided, and it can be printed.

A first order differential can be written as

$$\dot{x} = Ax \tag{20.0.1}$$

where A is an $n \times n$ matrix of coefficients for the vector of variables $\underset{n\times 1}{x}$.

Example 20.1. *Solution of $\ddot{y} + 4\dot{y} + 3y = 0$ using roots:*

$$\begin{aligned}
m^2 + 4m + 3 &= 0 \\
m &= -1, -3 \\
y_1(t) &= e^{-t} \\
y_2(t) &= e^{-3t} \\
W[y_1, y_2] &= \begin{vmatrix} e^{-t} & e^{-3t} \\ -e^{-t} & -3e^{-3t} \end{vmatrix} \\
&\neq 0 \\
\therefore y(t) &= c_1 e^{-t} + c_2^{-3t}
\end{aligned}$$

Now, converting it to a linear system:

$$\begin{aligned} x(t) &= \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \\ &= \begin{bmatrix} y \\ \dot{y} \end{bmatrix} \\ \dot{x}(t) &= \begin{bmatrix} x_2 \\ -4x_2 - 3x_1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

Without the use of Eigenvalues, this can be solved as:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} = \begin{bmatrix} c_1 y_1 + c_2 y_2 \\ c_1 \dot{y}_1 + c_2 \dot{y}_2 \end{bmatrix} = c_1 \begin{bmatrix} y_1 \\ \dot{y}_1 \end{bmatrix} + c_2 \begin{bmatrix} y_2 \\ \dot{y}_2 \end{bmatrix}$$

which means that the solution can be written as:

$$x(t) = c_1 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix} = c_1 x^{(1)}(t) + c_2 x^{(2)}(t)$$

notice that this is near identical to the Wronskian.

For the given differential equation

$$a\ddot{y} + b\dot{y} + cy = 0$$

the system of equations becomes:

$$\dot{x} = \begin{bmatrix} \dot{y} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} x_2 \\ -\frac{b}{a}x_2 - \frac{c}{a}x_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{20.0.2}$$

This can be solved by determining the eigenvalue λ and eigenvector ξ. To obtain the eigenvalue:

$$\det|\mathbf{A} - \lambda \mathbf{I}| = 0 \tag{20.0.3}$$

$$= \begin{vmatrix} -\lambda & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda \end{vmatrix} \tag{20.0.4}$$

$$\Rightarrow -\lambda\left(-\frac{b}{a} - \lambda\right) + \frac{c}{a} = 0 \tag{20.0.5}$$

$$\Rightarrow \lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \tag{20.0.6}$$

$$\lambda = \frac{-\frac{b}{a} \pm \sqrt{(\frac{b}{a})^2 - 4 \cdot \frac{c}{a}}}{2} \tag{20.0.7}$$

To obtain the eigenvectors $\xi^{(1)}$ and $\xi^{(2)}$,

$$(\mathbf{A} - \lambda_i \mathbf{I})\, \xi^{(i)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20.0.8}$$

$$\begin{bmatrix} -\lambda_i & 1 \\ -\frac{c}{a} & -\frac{b}{a} - \lambda_i \end{bmatrix} \begin{bmatrix} \xi_1^{(i)} \\ \xi_2^{(i)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{20.0.9}$$

$$-\lambda_i \xi_1^{(i)} + \xi_2^{(i)} = 0 \tag{20.0.10}$$

$$-\frac{c}{a}\xi_1^{(i)} + \left(-\frac{b}{a} - \lambda_i\right) \xi_2^{(i)} = 0 \tag{20.0.11}$$

Note that each solution has the format:

$$x^{(i)} = e^{\lambda_i t} \cdot \xi^{(i)} \tag{20.0.12}$$

and overall

$$x = c_1 \cdot x^{(1)} + c_2 \cdot x^{(2)} \tag{20.0.13}$$

Example 20.2. *Given*

$$A = \begin{bmatrix} 2 & 0 \\ 0 & -5 \end{bmatrix}$$

computing the eigenvalue ():

$$\det |\mathbf{A} - \lambda \mathbf{I}| = 0$$

$$\det \begin{vmatrix} 2 - \lambda & 0 \\ 0 & -5 - \lambda \end{vmatrix} = 0$$

$$\lambda^2 + 3\lambda - 10 = 0$$

$$\lambda = \frac{-3 \pm \sqrt{3^2 - 4 \cdot -10}}{2}$$

$$\lambda_1 = 2$$

$$\lambda_2 = -5$$

The corresponding eigenvectors is determined by:

$$\lambda_1 = 2$$

$$\xi^{(1)} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$(\mathbf{A} - \lambda_i \mathbf{I})\, \xi^{(1)} = \mathbf{0}$$

21 November 30: Homogenous Linear Systems with Real Eigenvalues

The definition of Eigenvalue λ and Eigenvector ξ is:

$$A\xi = \lambda\xi \tag{21.0.1}$$

The inverse of a two-by-two matrix takes the form:

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}. \tag{21.0.2}$$

Solving Homogenous Linear Systems with Real Eigenvalues

To compute the eigenvalue:

$$\det(\mathbf{A} - \lambda \mathbf{I}) = 0 \tag{21.0.3}$$

And once we get λ, we can find ξ using:

$$(\mathbf{A} - \lambda \mathbf{I})\xi = \mathbf{0} \tag{21.0.4}$$

A particular solution is given as:

$$x^{(i)} = e^{\lambda_i t}\xi \tag{21.0.5}$$

And the general solution is

$$x(t) = \sum c_i \cdot x^{(i)}(t) \tag{21.0.6}$$

Example 21.1. *Finding the eigenvalue and eigenvector for* **A**

$$\begin{aligned}
\mathbf{A} &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \\
\mathbf{A} - \lambda \mathbf{I} &= \begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= \lambda^2 + 1 \\
\det(\mathbf{A} - \lambda \mathbf{I}) &= 0 \\
\implies \lambda^2 + 1 &= 0 \\
\lambda &= \pm i \\
\lambda_1 &= i \\
\lambda_2 &= -i
\end{aligned}$$

Now, the eigenvectors:

$$\begin{aligned}
(\mathbf{A} - \lambda \mathbf{I})\xi &= 0 \\
\begin{bmatrix} -\lambda_1 & 1 \\ -1 & -\lambda_1 \end{bmatrix}\xi &= 0
\end{aligned}$$

22 December 5: Complex Eigenvalues

A complex eigenvalue is given with the notation

$$\lambda = \alpha \pm \beta i \tag{22.0.1}$$

in which case, the eigenvector can be written as

$$\xi = \vec{a} \pm \vec{b}i \tag{22.0.2}$$

The fundamental solution becomes the linear combination of particular solutions (just like 21.0.6), however, the particular solution becomes:

$$x^{(1)}(t) = e^{\alpha t}\left(\cos\beta t \cdot \vec{a} - \sin\beta t \cdot \vec{b}\right) \tag{22.0.3}$$

$$x^{(2)}(t) = e^{\alpha t}\left(\sin\beta t \cdot \vec{a} + \cos\beta t \cdot \vec{b}\right) \tag{22.0.4}$$

$$x(t) = c_1 x^{(1)}(t) + c_2 x^{(2)}(t) \tag{22.0.5}$$

Example 22.1. *Continuing from the previous example 21.1,*

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} \text{ for } \lambda_1 = i$$

And then for $\lambda_2 = -i$

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

Hence:

$$\begin{aligned} \xi &= \begin{bmatrix} 1 \\ \pm i \end{bmatrix} \\ &= \vec{a} \pm \vec{b}i \\ \vec{a} &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ \vec{b} &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{aligned}$$

This leads us to the particular solutions:

$$\begin{aligned} \alpha &= 0 \\ \beta &= 1 \\ x^{(1)}(t) &= e^{\alpha t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos\beta t - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin\beta t\right) \\ &= \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \\ x^{(2)}(t) &= e^{\alpha t}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin\beta t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos\beta t\right) \\ &= \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} \end{aligned}$$

A more practical example:

Example 22.2. *Given:*

$$\begin{aligned} \ddot{y} &= 2\dot{y} - 5y \\ \dot{x} &= \begin{bmatrix} x_2 \\ 2x_2 - 5x_1 \end{bmatrix} \\ &= Ax \\ A &= \begin{bmatrix} 0 & 1 \\ -5 & 2 \end{bmatrix} \end{aligned}$$

Find the eigenvalue and eigenvectors.

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= 0 \\ \begin{vmatrix} -\lambda & 1 \\ -5 & 2-\lambda \end{vmatrix} &= 0 \\ \lambda^2 - 2\lambda + 5 &= 0 \\ \lambda &= \frac{2 \pm \sqrt{4 - 4\cdot 5}}{2} \\ &= 1 \pm 2i \\ (\lambda_1, \lambda_2) &= (1+2i, 1-2i) \end{aligned}$$

Now, finding ξ:B

$$\begin{aligned} (\mathbf{A} - \lambda\mathbf{I})\xi &= \mathbf{0} \\ \begin{bmatrix} -1-2i & 1 \\ -5 & 2-1-2i \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \end{aligned}$$

Solving for $\xi^{(1)}$:

$$\begin{aligned} \xi_1^{(1)}(-1-2i) + \xi_2^{(1)} &= 0 \\ -5\xi_1^{(1)} + \xi_2^{(1)}(1-2i) &= 0 \\ -5\xi_1^{(1)} - \xi_1^{(1)}(-1-2i)(1-2i) &= 0 \\ -5\xi_1^{(1)} - \xi_1^{(1)}(-1 + \cancel{2i - 2i} - 2) &= 0 \\ \xi_1^{(1)} &= 0 \end{aligned}$$

To avoid getting a zero, we have to select an arbitrary non-zero value:

$$\begin{aligned} \xi_1^{(1)} &= 1 \\ \xi_2^{(1)} &= 1 + 2i \end{aligned}$$

Hence:

$$\xi = \begin{bmatrix} 1 \\ 1 \pm 2i \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \pm \begin{bmatrix} 0 \\ 2 \end{bmatrix} i = \vec{a} \pm \vec{b}i$$

This allows us to find the specific solutions:

$$\begin{aligned}
\alpha &= 1 \\
\beta &= 2 \\
x^{(1)}(t) &= e^{\alpha t}\left(\begin{bmatrix}1\\1\end{bmatrix}\cos\beta t - \begin{bmatrix}0\\2\end{bmatrix}\sin\beta t\right) \\
&= e^t\begin{bmatrix}\cos 2t\\ \cos 2t - 2\sin 2t\end{bmatrix} \\
x^{(2)}(t) &= e^{\alpha t}\left(\begin{bmatrix}1\\1\end{bmatrix}\sin\beta t + \begin{bmatrix}0\\2\end{bmatrix}\cos\beta t\right) \\
&= e^t\begin{bmatrix}\sin 2t\\ \sin 2t + 2\cos 2t\end{bmatrix} \\
x(t) &= c_1x^{(1)}(t) + c_2x^{(2)}(t)
\end{aligned}$$

Since

$$x(t) = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}y\\ \dot{y}\end{bmatrix}$$

We can write:

$$\begin{aligned}
y &= e^t(c_1\cos 2t + c_2\sin 2t) \\
\dot{y} &= e^t\left(c_1(\cos 2t - 2\sin 2t) + c_2(sin2t + 2\cos 2t)\right)
\end{aligned}$$

The benefit of this method is its ability to solve equations of arbitrary sizes.

23 December 7: Review for Final Exam

The Final will include chapter 3, 5, and 6. The following, in order of importance, should be reviewed for ideal preparation.

1. Midterm's problem 2, 3, 4, and 6 should be reviewed.
2. From the quiz, 5 to 9 should be reviewed.
3. Chapter 6 will have to be reviewed separately.
4. Review the notes.
5. Review outlines of chapter 5 and 6.
6. Do the Homework.

For the Final, make sure to:

1. Bring student ID
2. Bring **scientific** calculator

Quiz 9 has been graded, but to receive the paper, we will have to Email her and get it from her on Friday.

All of chapter 3 is about second order differential equation, chapter 5 is about Laplace transform, and chapter 6 is about using matrix math to solve it.

23.1 Review

1. The form of a 2nd order linear differential equation is

$$a_0(x)\ddot{y} + a_1(x)\dot{y} + a_2(x)y = g(x) \tag{23.1.1}$$

   and $a_0(x) \neq 0 \, \forall x$ and can be rewritten as

$$\ddot{y} + p(x)\dot{y} + q(x)y = f(x) \tag{23.1.2}$$

2. Integrating factor method does **not** work for the above.
3. The homogenous 2nd order linear differential equation can be solved with the superposition principle.
4. The superposition principle for homogenous equation is where there are two solutions for a linear differential equation: y_1 and y_2, and therefore:

$$y = c_1 y_1 + c_2 y_2 \tag{23.1.3}$$

   will also be a solution.
5. Initial Value Problem is where we are given $y(0)$ and $\dot{y}(0)$ to solve the problem. The initial values are **required** for solving using Laplace Transform.
6. A solution for a non-homogenous equation exists and is unique if $p(x), q(x), f(x)$ are all continuous for $x \in (a, b)$ with an initial condition existing in $x_0 \in (a, b)$; only applies to that interval.
7. y_1 and y_2 are linearly independent if

$$c_1 y_1 + c_2 y_2 = 0 \, \forall x \in x \tag{23.1.4}$$

   and the only solution is if c_1 and c_2 are 0. This can be confirmed using the Wronskian.
8. The Wronskian is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ \dot{y}_1 & \dot{y}_2 \end{vmatrix} \tag{23.1.5}$$

9. For the homogenous linear differential equation

$$\ddot{y} + p(x)\dot{y} + q(x)y = 0, \tag{23.1.6}$$

   the following are equivalent:

   (a) y_1 and y_2 are linearly independent solutions on $x \in (a, b)$
   (b) $W[y_1, y_2](x) \neq 0 \, \forall x \in (a, b)$
   (c) $W[y_1, y_2](x) \neq 0$ for at least one $x_0 \in (a, b)$

10. y_1 and y_2 are two linearly independent solutions (for homogenous) if any solution Y can be expressed as

$$Y(x) = c_1 y_1(x) + c_2 y_2(x) \tag{23.1.7}$$

11. For homogenous, y_1 and y_2 are the *fundamental* set of solutions, and the *general* solution is given as the linear combination of $y(x) = c_1 y_1(x) + c_2 y_2(x)$, which can be determined if we are given some initial conditions.
12. To solve the homogenous equation, knowing one solution $y_1(x)$, then we can solve for $y_2(x)$ even if $p(x), q(x)$ are functions of their own, by using the *Reduction of Order Method*, where

$$y_2(x) = v(x)y_1(x) \ni v(x) = \int \frac{e^{-\int p(s)ds}}{(y_1(x))^2} dx \tag{23.1.8}$$

note that the above integral is with respect to s because that is supposed to be done first, and then that can be changed to x.
13. For constant terms in $a\ddot{y} + b\dot{y} + cy = 0$, then $y(x) = e^{mx}$, where m can be obtained using

$$am^2 + bm + c = 0 \tag{23.1.9}$$

Depending on the discriminant $D = b^2 - 4ac$, if:

(a) $D > 0$, then we have two distinct roots r_1, r_2 and

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \tag{23.1.10}$$

(b) $D = 0$, then we have one root r, but

$$y(x) = c_1 e^{rx} + c_2 x e^{rx} \tag{23.1.11}$$

(c) $D < 0$, then our solution will have complex roots $p \pm qi$:

$$y(x) = e^{px}(c_1 \cos qx + c_2 \sin qx) \tag{23.1.12}$$

14. Consider the non-homogenous equation

$$\ddot{y} + p(x)\dot{y} + q(x)y = f(x) \tag{23.1.13}$$

then the general solution is

$$y(x) = y_h(x) + y_p(x) \tag{23.1.14}$$

where $y_h(x)$ is the general solution of the *homogenous* part, while the $y_p(x)$ is the particular non-homogenous solution part.
15. For non-homogenous, there also is a superposition principle, where $y(x)$ is the output (solution), and $f(x)$ is the non-homogenous part.

(a) Scalar property, where if $y(x) \longleftrightarrow f(x)$, then $ky(x) \longleftrightarrow kf(x)$
(b) Additive property, if $y_1 \longleftrightarrow f_1(x)$, and $y_2 \longleftrightarrow f_2(x)$, then $y_1 + y_2 \longleftrightarrow f_1 + f_2$

16. The particular solution y_p can be computed using:

    (a) Method of Undetermined Coefficient, which works if $f(x)$ is any of the following $x^n, e^{ax}, \sin\beta x, \cos\beta x$, or the products of those. In this case, use table 1 to choose a y_p, and then plug it into the original non-homogenous differential equation:

    $$\ddot{y}_p + p(x)\dot{y}_p + q(x)y_p = f(x) \tag{23.1.15}$$

    to determine the coefficients. Multiply with x^S, where S is the number of *non-homogenous* solutions that the particular solution candidate matches with. $f(x)$ can be the product of multiple candidates, in which case, we can exclude any *additional* coefficient A.

    (b) Method of Variation of Parameters, where:

    $$y_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x) \tag{23.1.16}$$

    in which y_1 and y_2 are the fundamental set of solutions for the *homo*genous part of the solution. v is gives with:

    $$v_1' = -\frac{y_2 f}{W(y_1, y_2)} \tag{23.1.17}$$

    $$v_2' = \frac{y_1 f}{W(y_1, y_2)} \tag{23.1.18}$$

    and

    $$v_1' y_1 + v_2' y_2 = 0 \tag{23.1.19}$$

    $$v_1' y_1' + v_2' y_2' = f(x) \tag{23.1.20}$$

24 December 13: Planning for Extra Credit

Please write a one-page report (single-sided) about the applications of differential equations (where you can use differential equations). You can use some references, but all sentences need to be written as your own and paraphrased. Also, the examples or applications should be outside of the textbook. You may be able to include some equations and explain them as an example. The report needs to be single-spaced with 11-point font as a Docx or PDF file.

1. What differential equations are.
2. Summary of applications.
3. Logarithmic growth $\frac{dP}{dt} = kp(1 - \frac{P}{L})$

4. KVR equation in Unit 23 RLC Circuits pre-lecture
5. My research with non-linear, sine.
6. Summarize.