ENME 221: Dynamics

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Contents

Our TA is Conor Fitzpatrick. Our TFs are Gabe, Katie, August, and Darryl. We will be doing a mixed bag of online and in-person office hours. Our professor is Dr. Joshua Radice, email is radice@usna.edu. For online office hours, it will be on Google Meet.

Category	%
Exams (4)	80
Quizzes and Homework	20

Table 1: Grading Scheme

A lot of example problems will be given throughout the semester. We will be sending scans of our homework online. We will not have common final. The exams are 'accidentally' cumulative, as in we will use old material.

The next assignment is already released. The quizzes we will have (they will be surprises) will be from the homework, hence recommended doing the homework. We can work together. Exams are always 3 questions. One of the exam problems will be from homework. Homework due Sunday night. We should do one problem in each page. In the exam, only one question will have algebra that we need to solve.

We don't need to do algebra, as in we can use things like MATLAB to solve equations. We need our final answer, if possible, to be in a closed equation. A day by day schedule is provided in the syllabus. We will not have class next Friday, since the professor has something at his main job.

In discussion, it will be like an open forum. The last 20 minutes will be a quiz. We can use the equation sheet.

Questions to Ask:

- ☑ Grading percentage *between* quizzes and homework? A ratio that is most advantageous.
- ☑ Can we use 8th edition of the book? Will not have proper homework questions, so no.
- ☑ Will the equation sheet be on the exams? Yes.

2/01: Rectilinear Kinematics

In this class, we will not consider deformation, except for springs. This class will have two main divisions:

1. Kinematics: Pure description/study of motion (e.g. projectile motion).
2. Kinetics: Loads and the motion they cause.

A particle is a point mass. A problem should be dealt with like a point mass if the dimension does not matter. Rigid bodies should be used if dimensions and rotations matter. The four quadrants are:

	Particle	Rigid Bodies
Kinematics	Exam 1	Exam
Kinetics	Exam	Exam 4

Table 2: Four Quadrants of Exams

We will focus on particle kinematics for now. A rectilinear description is a *scalar* description that describes what a particle is doing, along its flight path. Example: a roller coaster that only knows to accelerate the car and path length.

s is the arc length along a given path (position). Our displacement is given as

$$\Delta s = s_2 - s_1 \tag{1}$$

The average speed is given as

$$v_{avg} = \frac{\Delta s}{\Delta t} \tag{2}$$

While the instantaneous speed is given as

$$v = \frac{\mathrm{d}s}{\mathrm{d}t} \tag{3}$$

The average acceleration is given as

$$a_{avg} = \frac{\Delta v}{\Delta t} \tag{4}$$

and the instantaneous acceleration is

$$a_t = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}^2 s}{\mathrm{d}t^2} \tag{5}$$

Note that acceleration can be a function of the position, or also velocity (e.g. acceleration decreases when speed increases (to reach terminal velocity)). Therefore:

$$\begin{aligned} \frac{\mathrm{d}s}{\mathrm{d}t} &= v \\ \frac{\mathrm{d}v}{\mathrm{d}t} &= a \\ \frac{\mathrm{d}s}{v} &= \frac{\mathrm{d}v}{a} \\ \therefore a_t \, \mathrm{d}s &= v \, \mathrm{d}v \end{aligned} \tag{6}$$

Example 1

A particle travels in straight line with equation

$$v = \left(\frac{t^3}{2} - 8t\right)\left[\frac{m}{s}\right]$$

The particle starts at $s = 1$. Find:

- ☐ acceleration at 2 seconds $a_t(2)$,
- ☐ when $a(t_f) = 0$,
- ☐ where the particle is at $t = 3$, that is, $s(3)$.

For the first problem:

$$\begin{aligned} \dot{v} &= \frac{3}{2}t^2 - 8\left[\frac{m}{s}\right] \\ \therefore a(2) &= -2\frac{m}{s^2} \end{aligned}$$

For the second:

$$\begin{aligned} \frac{3}{2}t^2 - 8 &= 0 \\ t_f &= \sqrt{8 \cdot \frac{2}{3}} \\ &= 2.31s \end{aligned}$$

And finally, take the integral for the position:

$$\begin{aligned} s &= \int \frac{t^3}{2} - 8t \, \mathrm{d}t \\ &= \frac{t^4}{8} - 4t^2 + C_1\,[m] \\ s(0) &= 1\,m \\ s &= \frac{t^4}{8} - 4t^2 + 1\,[m] \\ s(3) &= -24.875\,m \end{aligned}$$

A dot denotes time differentiation (e.g. $\dot{v}$), while a comma denotes space differentiation (v'). The professor does not care about how many significant or decimal places we use.

Example 2

From what floor of a building must a car be dropped from an at-rest position so that it reaches a speed of 80.7 ft/s (55 mph) when it hits the ground? (Note: each floor is 12 ft.) How long does it take to reach the ground?

Considering origin to be the floor, with positive direction being up:

$$\begin{aligned} a_t &= -g \\ v &= -gt + \cancelto{0}{C_1} \\ s &= -\frac{gt^2}{2} + H \\ s(t_f) &= -\frac{gt_f^2}{2} + H \\ &= 0 \\ v(t_f) &= -gt_f \\ &= -80.7 \\ \therefore t_f &= 2506s \\ \therefore H &= 101.13ft \end{aligned}$$

Calculators are allowed in the exam, but only one problem is numerical.

2/03: Curvilinear Motion and Cartesian Coordinates

When we are integrating with respect to anything, there will be a constant of integration.

To solve cases where the function is not directly in respect to time, we can do:

$$a\,\mathrm{d}s = v\,\mathrm{d}v \tag{7}$$

Examples when this is the case are spring acceleration (dependent on position), and drag which is a function of velocity

$$a\,\mathrm{d}v = v\,\mathrm{d}v \tag{8}$$

Example 3

A projectile is pushed through along a magnetic track that imparts an acceleration according to the following profile:

$$a = (8 - 2s)\left[\frac{m}{s^2}\right]$$

If the projectile is initially at rest, what is the speed of the particle as it exits the 4m long magnetic track?

Given:

$$\begin{aligned} a &= 8 - 2s \\ v(0) &= 0 \\ L &= 4\,\mathrm{m} \end{aligned}$$

Asked for $v(L)$. Solution:

$$\begin{aligned} a\,\mathrm{d}s &= v\,\mathrm{d}v \\ \int 8 - 2s\,\mathrm{d}s &= \int v\,\mathrm{d}v \\ 8s - s^2 + C_1 &= \frac{v^2}{2} \\ v &= \sqrt{16s - 2s^2 + C_2} \end{aligned}$$

Then, inserting the initial condition,

$$\begin{aligned} v(0) &= \sqrt{C_2} \\ v &= \sqrt{16s - 2s^2}\ [\mathrm{m/s}] \\ v(L) &= \sqrt{16 \times 4 - 2 \times 4^2} \\ &= \boxed{5.66\ \mathrm{m/s}} \end{aligned}$$

A rectilinear description of motion is the scalar description along the flight path. A curvilinear description is a *vector* description. In the latter, $\vec{r}$ denotes the position vector. In this case, the displacement is given as:

$$\Delta\vec{r} = \vec{r}_2 - \vec{r}_1 \tag{9}$$

and the average velocity is:

$$\vec{v}_{avg} = \frac{\Delta\vec{r}}{\Delta t} \tag{10}$$

Instantaneous is

$$\vec{v} = \frac{\mathrm{d}\vec{r}}{\mathrm{d}t} \tag{11}$$

The average is the secant, the instantaneous is tangent.

Similar applies to acceleration:

$$\vec{a}_{avg} = \frac{\Delta\vec{v}}{\Delta t} \tag{12}$$

$$\vec{a} = \frac{\mathrm{d}\vec{v}}{\Delta t} \tag{13}$$

Acceleration can change our speed, our direction, or both.

The derivative of the product of scalar functions is

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)\beta(t) = \dot{\alpha}\beta + \alpha\dot{\beta}$$

and similarly, the derivative of the product of scalar and vector is

$$\frac{\mathrm{d}}{\mathrm{d}t}\alpha(t)\vec{A}(t) = \dot{\alpha}\vec{A} + \alpha\dot{\vec{A}} \tag{14}$$

For strict vectors

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{A}(t)\cdot\vec{B}(t) = \dot{\vec{A}}\cdot\vec{B} + \vec{A}\cdot\dot{\vec{B}} \tag{15}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{A}(t)\times\vec{B}(t) = \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}} \tag{16}$$

In Cartesian coordinates, the position vector can be denoted in $\hat{i}, \hat{j}, \hat{k}$ format:

$$\vec{r} = r_x\hat{i} + r_y\hat{j} + r_z\hat{k} \tag{17}$$

Note that when taking derivative of the position vector, we also take the derivatives for $\hat{i}, \hat{j}, \hat{k}$, but those turn out to be 1 since they are constant.

$$\dot{\vec{r}} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \tag{18}$$
$$= \vec{v} \tag{19}$$
$$\ddot{\vec{r}} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} \tag{20}$$
$$= \vec{a} \tag{21}$$

To obtain the direction only, we can obtain a unit vector

$$\hat{u}_T = \frac{\vec{v}}{|\vec{v}|} \tag{22}$$

which varies over time. The magnitude of the velocity is speed, but the magnitude of the acceleration is not too useful because there is no useful physical quantity expression for it.

Example 4

The velocity of a particle is

$$\hat{v} = \langle 2+t, 4t-2t^2, t-1 \rangle \text{ [m/s]}$$

is initially at the origin. Determine the speed of the particle as a function of time. Determine the acceleration of the particle as a function of time. Determine the displacement of the particle between 2s and 3s. Asked to find

- ☐ $v(t)$
- ☐ $a(\vec{t})$
- ☐ $\Delta\vec{r} = \vec{r}(3) - \vec{r}(2)$

For the speed,

$$v(t) = \sqrt{(2+t)^2 + (4t-2t^2)^2 + (t-1)^2} \text{ [m/s]}$$

and for the acceleration (vector),

$$\vec{a}(t) = \langle 1, 4-4t, 1 \rangle \text{ [m/s}^2\text{]}$$

For the displacement, first obtain the equation:

$$\vec{r}(t) = \int \vec{v}\,\mathrm{d}t$$
$$= \langle 2t + \frac{1}{2}t^2,\ 2t^2 - \frac{2}{3}t^3,\ \frac{1}{2}t^2 - t \rangle$$

And then just find the difference.

Important note: the braket notation is **not** recommended due to ambiguity.

Example 5

The particle starts at the origin and travels along the path defined by the parabola $y = 0.5x^2$. The component of velocity along the x axis is $v_x = (0.5t)$ ft/s, where t is in seconds. Determine the particle's distance from the origin and the magnitude of the particle's acceleration one second later.

2/18: Asynchronous Class

Steps for doing pully problems:

1. Determine the number of *independent* working cables
2. Define moving points or bodies in respect to fixed axis
3. Define/specify positive directions
4. Write working cable length equations in terms of moving particle positions
5. Assuming fixed length cables, take the derivative of the length in respect to time
6. Take the derivative another time for acceleration
7. Solve for the missing variables

2/15: The normal-tangental coordinate system

Positive is generally taken *away* from the origin, or points that does not change.

Unlike the Cartesian coordinate system that is fixed in space, the normal-tangental coordinate system follows the point. It involves the tangent, normal, and binormal unit vectors:

$$\hat{u}_T,\ \hat{u}_N,\ \hat{u}_B \tag{23}$$

The tangent is the direction of travel of the particle, whereas the normal unit vector points to the direction of turning. The normal vector is always perpendicular to the tangent vector.

$$\hat{u}_B = \hat{u}_T \times \hat{u}_N \tag{24}$$

The origin, in this case, does not matter. The velocity will always be the scalar speed in our direction of travel

$$\vec{v} = v\hat{u}_T \tag{25}$$

The acceleration is given as

$$\vec{a} = \dot{v}\hat{u}_T + v\dot{\hat{u}}_T$$

Taking the infinitesimal change in time, we get a radius of curvature ρ:

$$\begin{aligned} \mathrm{d}s &= \rho\,\mathrm{d}\theta \\ \mathrm{d}\hat{u}_T &= \mathrm{d}\theta\hat{u}_N \\ \dot{\hat{u}}_T &= \frac{\mathrm{d}s}{\rho}\hat{u}_N \\ \dot{\hat{u}}_T &= \frac{v}{\rho}\hat{u}_N \end{aligned}$$

therefore the acceleration is

$$\vec{a} = \dot{v}\hat{u}_T + \frac{v^2}{\rho}\hat{u}_N \tag{26}$$

Example 6

If the car decelerates uniformly along the curved road from 20 m/s at A to 15 m/s at C, determine the acceleration of the car at B.

Given $s_{AB} = 250\,\mathrm{m}$, $s_{BC} = 50\,\mathrm{m}$, $v_A = 20\,\mathrm{m/s}$, $v_C = 15\,\mathrm{m/s}$, $\rho_B = 300\,\mathrm{m}$.

$$\begin{aligned} \vec{a}_B &= \dot{v}_B\hat{u}_T + \frac{v_B^2}{\rho_B}\hat{u}_N \\ a_T\,\mathrm{d}s &= v_T\,\mathrm{d}v \\ \Rightarrow a_T s + C &= \frac{v^2}{2} \text{ (integration)} \\ C &= \frac{v_A^2}{2} - a_T\cdot 0 \\ &= 200\,\mathrm{m^2/s^2} \\ \Rightarrow a_T &= \frac{1}{s_{AB}+s_{BC}}\left(\frac{v_B^2}{2} - C\right) \\ &= -0.291\bar{6}\,\mathrm{m/s^2} \\ v_B &= \sqrt{2a_T\cdot s_{AB} + 2C} \\ &= 15.943\,\mathrm{m/s} \\ a_N &= \frac{v_B^2}{\rho} \\ &= 0.847\bar{2}\,\mathrm{m/s} \\ \therefore \vec{a}_B &= -0.291\bar{6}\hat{u}_T + 0.847\bar{2}\hat{u}_N\ \left[\mathrm{m/s^2}\right] \end{aligned}$$

Uniform acceleration is about speed, not about change in direction, normally.

2/17: N-T and Polar Coordinates

From last class, we learned that N-T coordinates care about where we are going and turning, not where we are. The tangent points to where we are going, and the normal direction points where we are turning. The binormal is their cross product.

$$\vec{v} = v\hat{u}_T \tag{27}$$

$$v = |\hat{v}| \tag{28}$$

$$\vec{a} = \dot{v}\hat{u}_T + \frac{v^2}{\rho}\hat{u}_N \tag{29}$$

$$|\vec{a}| = \sqrt{\dot{v}^2 + \left(\frac{v^2}{\rho}\right)^2} \tag{30}$$

In our example problems, if we are asked to find the acceleration, find the vector.

Example 7

When a car starts to round a curved road with the radius of curvature of 600ft, it is traveling at 75 ft/s. The car's speed begins to decrease at a rate of $-0.006t^2\,\mathrm{ft/s^2}$. Determine and draw the acceleration of the car when it has traveled a distance of 700ft.

Given:

$$\begin{aligned} v_0 &= 75\,\mathrm{ft/s} \\ \rho &= 600\,\mathrm{ft} \\ L &= 700\,\mathrm{ft} \\ \dot{v} &= -0.006t^2\ \left[\mathrm{ft/s^2}\right] \\ &= -at^2 \end{aligned}$$

We use N-T coordinates because the car is moving in a path that is not so clear. We know the relationship that

$$\begin{aligned} \vec{a}_f &= \dot{v}_f\hat{u}_T + \frac{v_f^2}{\rho}\hat{u}_N \\ \dot{v} &= -at^2 \end{aligned}$$

We should think about exactly what we will do before actually do it. We double integrate the $\dot{v}$:

$$\begin{aligned} v &= -\frac{at^3}{3} + v_0 \\ s &= -\frac{at^4}{12} + v_0 t + \cancelto{0}{s_0} \\ s(t_f) &= L \\ \Rightarrow -\frac{at^4}{12} + v_0 t &= L \\ \Rightarrow t_f &= 9.38\,\text{s} \\ v(t_f) &= 73.34\,\text{ft/s} \\ \dot{v}(t_f) &= -at_f^2 \\ \vec{a} &= 0.528\hat{u}_T + 8.966\hat{u}_N \ \left[\text{ft/s}^2\right] \end{aligned}$$

In our homework problem, we must define the base vectors graphically (on the figure). We should also draw the acceleration and velocity vectors.

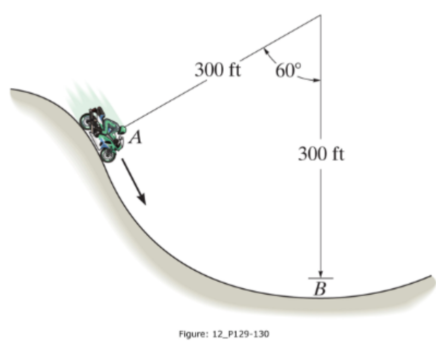

Figure 1: Example 8 Figure

Example 8

The motorcyclist starts at A with a speed of 3 ft/s and increases her speed at a rate of $(0.1s)\,\text{m/s}^2$ where s is measured in ft. Determine and sketch the velocity and acceleration when the motorcycle reaches point B.

We are given:

$$\begin{aligned} \dot{v} &= a_T \\ &= 0.1s\ \left[\text{m/s}^2\right] \\ &= as \\ R &= 300\,\text{ft} \\ \theta &= 60^\circ \\ v_0 &= 3\,\text{ft/s} \end{aligned}$$

and asked to find $\vec{v}, \vec{a}$. Like previous, our relationships are

$$\begin{aligned} \vec{a}_f &= \dot{v}_f \hat{u}_T + \frac{v_f^2}{\rho}\hat{u}_N \\ \dot{v} &= -at^2 \end{aligned}$$

We can find the arc length to find the distance covered (note, θ is used in radians):

$$\begin{aligned} s_f &= R\theta \\ &= 300\,\text{ft} \times \frac{60^\circ}{180^\circ}\pi\,\text{rad} \\ &= 314.159\,\text{ft} \end{aligned}$$

and therefore

$$\begin{aligned} \dot{v}_f &= 31.416\,\text{ft/s}^2 \\ a_t\,\text{d}s &= v\,\text{d}v \\ \int as\,\text{d}s &= \int v\,\text{d}v \\ \frac{as^2}{2} + C_1 &= \frac{v^2}{2} \\ v &= \sqrt{as^2 + C_1} \\ v_0 &= \sqrt{C_1} \\ \Rightarrow C_1 &= v_0^2\ \left[\text{ft}^2/\text{s}^2\right] \end{aligned}$$

Now find the values

$$\begin{aligned} v &= \sqrt{as^2 + v_0^2} \\ \Rightarrow v_f &= 99.39\,\text{ft/s} \\ \therefore \vec{v} &= 99.39\hat{u}_T\ \text{ft/s} \\ \therefore \vec{a} &= 31.4\hat{u}_T + 32.92\hat{u}_N\ \left[\text{ft/s}^2\right] \end{aligned}$$

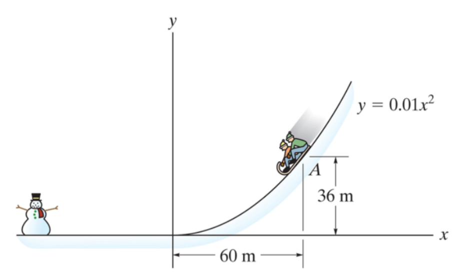

Figure 2: Example 9

Example 9

A toboggan is traveling down a curve which can be approximated by the equation shown. Determine and draw the acceleration at point A where it's speed is 10m/s and it is increasing at a rate of 3 m/s^2

In this problem, referring to 2, our tangent direction is down the hill along the path of travel, and normal vector is to the left.

$$\begin{aligned}\vec{v} &= v\hat{u}_T \\ &= 10\hat{u}_T\,\text{m/s}\end{aligned}$$

The equation for a complicated ρ formula is given in our equation sheet, about 2/3rd the way down:

$$\rho = \frac{\left(1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2\right)^{\frac{3}{2}}}{\frac{\mathrm{d}^2y}{\mathrm{d}x^2}} \tag{31}$$

The acceleration vector will be to the left, since some of the acceleration is to the top left, and some to down left.

We will focus on cylindrical polar coordinates. When we describe cylindrical polars, we describe using r and θ; the third direction is simply z. Spherical polars are more challenging, and we will not be using them yet; they can be more practical though, such as in the case of our eyes. Our ears perceive the world N-Ts. We get motion sick because eyes and ears give two different bits of information.

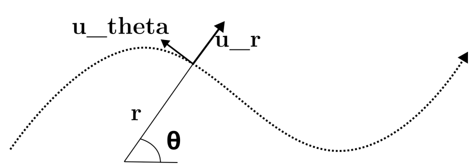

Figure 3: Cylindrical Polar Coordinates

Our $\hat{u}_\theta$ has nothing to do with motion of particle. It only relates with our position. It is 90○ counterclockwise to our $\hat{u}_r$. Our $\hat{u}_r$ points away from the origin.

If it is circular motion, the $\hat{u}_r$ points away from the circle, while $\hat{u}_N$ points in. The $\hat{u}_\theta$ and $\hat{u}_T$ also point in opposite direction.

One of the problems is that, in motion, both of the polar coordinate vectors change. We define a position vector

$$\vec{r} = r\hat{u}_r \tag{32}$$

and therefore

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \dot{r}\hat{u}_r + r\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}t} \tag{33}$$

2/22: C-P Derivations and Problems

Cylindrical coordinates is the hardest. In this class, we learned about rectilinear and N-T description.

The two unit vectors for cylindrical polar are the $\hat{u}_\theta$ and $\hat{u}_r$ (and the z). This is different from N-T. r does not depend on what the particle is doing; just where it is (In contrast, $\hat{u}_N$ is curvature). The r is just the direction from where the particle is from the viewer/origin. $\hat{u}_\theta$ is just 90 degrees counter clockwise from $\hat{u}_r$. There is an origin in this case, and a position vector:

$$\vec{r} = r\hat{u}_r \tag{34}$$

and the velocity vector is

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{v} = \dot{r}\hat{u}_r + r\frac{\mathrm{d}\hat{u}_r}{\mathrm{d}t} \tag{35}$$

When a particle moves, the particle experiences a change in angle $\mathrm{d}\theta$.

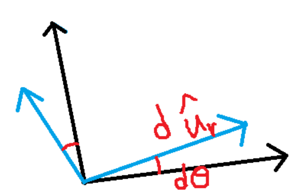

Figure 4: Deriving the velocity vector

Referring to figure 4, where the blue line denotes the changed position, we can write our velocity vector 35 as

$$\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\,\hat{u}_\theta. \tag{36}$$

A real world example is where a circular disk spins, and its angular velocity is basically $\dot{\theta}$. ω will be a specific property on a rigid body: if one particle on a body has an ω, another point also has it.

Taking the derivative of 36, we have:

$$\frac{\mathrm{d}\vec{v}}{\mathrm{d}t} = \vec{a} \tag{37}$$

$$= \ddot{r}\hat{u}_r + \dot{r}\dot{\hat{u}}_r + \dot{r}\dot{\theta}\hat{u}_\theta + r\ddot{\theta}\hat{u}_\theta + r\dot{\theta}\dot{\hat{u}}_\theta \tag{38}$$

$$\frac{\mathrm{d}\hat{u}_\theta}{\mathrm{d}t} = -\frac{\mathrm{d}\theta}{\mathrm{d}t} \tag{39}$$

$$= -\dot{\theta}\hat{u}_r \tag{40}$$

$$\implies \vec{a} = \ddot{r}\hat{u}_r + 2\dot{r}\dot{\theta}\hat{u}_\theta + r\ddot{\theta}\hat{u}_\theta - r\dot{\theta}^2\hat{u}_r \tag{41}$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta \tag{42}$$

where the first term for $\hat{u}_r$ is how much the point is running away from the origin. The second term (for $\hat{u}_r$) is the centripetal acceleration:

$$a_c = R\omega^2 = r\dot{\theta}^2 \tag{43}$$

In the second term, the $r\ddot{\theta}$ is the $R\alpha$ term we'd see in Physics. The $2\dot{r}\dot{\theta}$ is the Coriolis effect. These are for equation 42.

Also, note that the in the derivation:

$$2\dot{r}\dot{\theta}\hat{u}_\theta = \dot{r}\dot{\hat{u}}_r + \dot{r}\dot{\theta}\hat{u}_\theta$$

If we are the particle, then we should describe using N-T coordinates. If we are the observer, then we should use cylindrical polar (C-P).

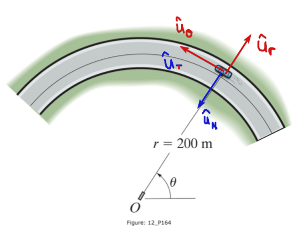

Figure 5: Example 10

Example 10

A radar gun at O rotates with the angular velocity of 0.1 rad/s and an angular acceleration of 0.025 rad/s2 at the instant the angle is 45°. It follows the car as it drives a circular road with a radius of 200m. Determine and draw the velocity and acceleration at this point.

Given:

$$\dot{\theta} = 0.1\,\mathrm{rad/s}$$
$$\ddot{\theta} = 0.025\,\mathrm{rad/s^2}$$
$$r = 200\,\mathrm{m}$$
$$\theta = 45\,^\circ\mathrm{C}$$

Because this uses angles and angular units, we should use C-P coordinates.
We are asked to find $\vec{v}$ and $\vec{a}$.
$\hat{u}_r$ points away from the origin. Note that in this case, looking at 5, the $\hat{u}_T$ and $\hat{u}_\theta$ just coincidentally happen to be in the same direction. This isn't always the case.

$$\vec{v} = \cancelto{0}{\dot{r}}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$$
$$= 20\hat{u}_\theta\,\mathrm{m/s}$$

$$\vec{a} = \left(\cancelto{0}{\ddot{r}} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + \cancelto{0}{2\dot{r}\dot{\theta}}\right)\hat{u}_\theta$$
$$= -2\hat{u}_r + 5\hat{u}_\theta\,\mathrm{m/s^2}$$

When we do these problems, we must draw the vectors to show that we understand. The result is shown in 6.

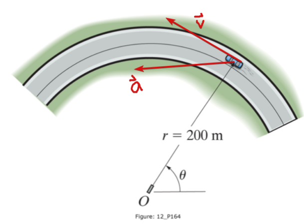

Figure 6: Solution drawing to example 10

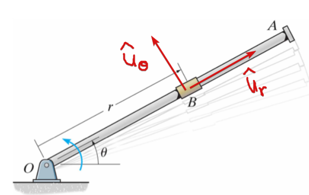

Figure 7: Example 11

Example 11

Rod OA is rotating counterclockwise with an angular velocity of 2t 2 rad/s. Through mechanical means, collar B moves outward from O along the rod at a speed of 4t 2 m/s. If θ and r are initially zero, determine the velocity and acceleration when the angle is 60°.

We are given

$$\begin{aligned}\dot{\theta} &= 2t^2\,\text{rad/s}\\ \dot{r} &= 4t^2\,\text{m/s}\\ \theta_0 &= 0\\ r_0 &= 0\\ \theta_f &= 60^\circ\\ &= \frac{\pi}{3}\,\text{rad}\end{aligned}$$

Problems like this should be solved using C-P. Use radians instead of degrees.
We are asked to find $\vec{v}$ and $\vec{a}$.

$$\begin{aligned}\vec{v}_f &= \dot{r}_f\hat{u}_r + r_f\dot{\theta}_f\hat{u}_\theta\\ \vec{a} &= \left(\ddot{r}_f - r_f\dot{\theta}_f^2\right) + \left(r_f\ddot{\theta}_f + 2\dot{r}_f\dot{\theta}_f\right)\hat{u}_\theta\end{aligned}$$

We derive or take integral of the givens:

$$\begin{aligned}r &= \frac{4}{3}t^3 + \cancel{C}^{\,0}\\ \dot{\theta} &= 2t^2\,\text{rad/s}\\ \dot{r} &= 4t^2\,\text{m/s}\\ \ddot{\theta} &= 4t\,\text{rad/s}\\ \ddot{r} &= 8t\,\text{m/s}\end{aligned}$$

Now placing this into our equations for $\vec{v}$ and $\vec{a}$

$$\begin{aligned}\vec{v} &= 4t^2\hat{u}_r + \frac{4t^3}{3}\cdot 2t^2\hat{u}_\theta\\ \vec{a} &= \left(8t - \frac{4t^3}{3}\cdot\left(2t^2\right)^2\right)\hat{u}_r + \left(\frac{4t^3}{3}\cdot 4t + 2\cdot 4t^2\cdot 2t^2\right)\hat{u}_\theta\end{aligned}$$

Evaluating these equations at $t_f = 1.16\,\text{s}$, we get

$$\begin{aligned}\vec{v} &= 5.41\hat{u}_r + 5.66\hat{u}_\theta\ [\text{m/s}]\\ \vec{a} &= -6\hat{u}_r + 38.95\hat{u}_\theta\ \left[\text{m/s}^2\right]\end{aligned}$$

The drawing of the solution is shown in 6.

We will later learn about transforming the unit

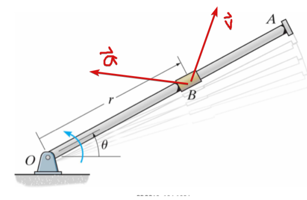

Figure 8: Example 11 Solution

vectors. So far, we have not used the z component for the C-P coordinates.

2/24: Using Different Coordinate Systems

The professor's office hours is actually before class, and he generally comes at 0745. He will update the syllabus.

In last class, we talked about different ways to describe motion. The N-T coordinate system is in perspective with the particle,

$$\vec{v} = v\hat{u}_T, \tag{44}$$

while C-P is from the perspective of an observer,

$$\vec{v} = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta. \tag{45}$$

Today, we will learn about using different coordinate systems depending on what is best, and also how to link them together.

To link the different coordinate systems together, we use cartesian systems.

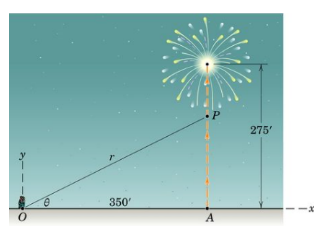

Figure 9: Example 12

Example 12

A fireworks shell is launched upward from point A and reaches its apex at an altitude of 275ft. Determine the quantities for when the shell

$$r, \dot{r}, \ddot{r}, \theta, \dot{\theta}, \ddot{\theta}$$

is at an altitude of 175ft

We are given

$$H = 275\,\text{ft}$$
$$h = 175\,\text{ft}$$
$$L = 350\,\text{ft}$$
$$g = 32.2\,\text{ft/s}^2$$

We will do a rectilinear description first. We can integrate the acceleration to determine the speed and height. We know that speed at top most position is zero.

$$\begin{aligned}
v\,\mathrm{d}v &= a\,\mathrm{d}s \\
&= -g\,\mathrm{d}s \\
\Rightarrow \int v\,\mathrm{d}v &= \int -g\,\mathrm{d}s \\
\frac{v^2}{2} &= -gs + C_1 \\
v &= \sqrt{-2gs + C_2} \\
v\big|_0 &= \sqrt{C_2} \\
&= v_0 \\
v &= \sqrt{-2gs + v_0} \\
\Rightarrow 0 &= \sqrt{-2gH + v_0^2} \\
\Rightarrow v_0 &= \sqrt{2gH} \\
\Rightarrow v &= \sqrt{-2gs + 2gH} \\
v(h) &= \sqrt{-2gh + 2gH} \\
&= v_p
\end{aligned}$$

Now that we have a rectilinear description of motion, we can convert it to other forms. Since the tangent direction is up, and it is going straight, we get and convert:

$$\begin{aligned}
\vec{v} = v_p \hat{u}_T \quad &= \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta \\
\vec{a} = -g\hat{u}_T \quad &= (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + (r\dot{\theta} + 2\dot{r}\dot{\theta})\hat{u}_\theta
\end{aligned}$$

If θ goes to zero, the $\hat{u}_r$ is in the same direction as $\hat{\imath}$, but slowly decreases. Hence, it is cosine. The same technique can be used to determine the conversions for the other components:

$$\begin{aligned}
\hat{u}_r &= (\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath} \\
\hat{u}_\theta &= (-\sin)\hat{\imath} + (\cos\theta)\hat{\jmath} \\
\hat{u}_T &= 0\hat{\imath} + 1\hat{\jmath}.
\end{aligned}$$

We are using Cartesian coordinates (the right hand side in the above equations), N-T coordinates for the fireworks and C-P for our question. Plugging the above equations into that for $\vec{v}, \vec{a}$ for C-P:

$$\begin{aligned}
\vec{v} = v_p\hat{\jmath} \quad &= \dot{r}\left[(\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath}\right] + r\dot{\theta}\left[(-\sin)\hat{\imath} + (\cos\theta)\hat{\jmath}\right] \\
\vec{a} = -g\hat{\jmath} \quad &= (\ddot{r} - r\dot{\theta}^2)\left[(\cos\theta)\hat{\imath} + (\sin\theta)\hat{\jmath}\right] \\
&\quad \ldots + (r\dot{\theta} + 2\dot{r}\dot{\theta})\left[(-\sin)\hat{\imath} + (\cos)\hat{\jmath}q\right]
\end{aligned}$$

We have four equations (v_i, v_j, a_i, a_j) from above, but 6 unknowns to find. However, we can use basic trigonometry to get two more equations:

$$\begin{aligned}
r &= \sqrt{h^2 + L^2} \\
\theta &= \arctan\left(\frac{h}{L}\right)
\end{aligned}$$

We will not be asked questions like this on the exam, since solving this will require something like MATLAB.

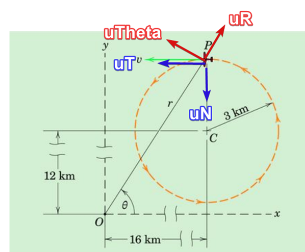

Figure 10: Example 14

Example 14

The low flying aircraft is traveling at a constant speed of 360kph in the holding pattern with a radius of 3km. Determine the quantities for

$$r, \dot{r}, \ddot{r}, \theta, \dot{\theta}, \ddot{\theta}$$

at the instant shown.

We are given:

$$\begin{aligned}
v &= 360 \times \frac{1000}{3600}\,\text{m/s} \\
\dot{v} &= 0 \\
\rho &= 3000\,\text{m} \\
X &= 16000\,\text{m} \\
Y &= 12000\,\text{m}
\end{aligned}$$

Even if we are tempted to, we should not alter the $\hat{i}, \hat{j}$ coordinate directions. Our main equations are:

$$\vec{v} = v\hat{u}_T \qquad = \dot{r}\hat{u}_r + r\dot{\theta}\hat{u}_\theta$$
$$\vec{a} = \frac{v^2}{\rho}\hat{u}_N \qquad = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta$$

and in cartesian:

$$\hat{u}_T = -1\hat{i} + 0\hat{j}$$
$$\hat{u}_N = 0\hat{i} - 1\hat{j}$$
$$\hat{u}_r = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$
$$\hat{u}_\theta = (-\sin\theta)\hat{i} + (\cos\theta)\hat{j}$$

Plugging it in separately:

$$v_i: \quad -v = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$
$$v_j: \quad 0 = \dot{r}\sin\theta + r\dot{\theta}\cos\theta$$
$$a_i: \quad 0 = (\ddot{r} - r\dot{\theta}^2)\cos\theta - (r\ddot{\theta} + 2\dot{r}\dot{\theta})\sin\theta$$
$$a_j: \quad -\frac{v^2}{\rho} = (\ddot{r} - r\dot{\theta}^2)\sin\theta + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\cos\theta$$

and the two additional geometric equations:

$$r = \sqrt{X^2 + (Y+\rho)^2}$$
$$\theta = \arctan\left(\frac{Y+\rho}{X}\right)$$

The professor has not shared the MATLAB script for solving system of equations, but will.

3/01: Relative Motion

Per the syllabus, we will be talking about relative motion. We have an exam on Wednesday, a week from today:

- Rectilinear motion
- N-T
- C-P
- Pulleys
- Potentially relative motion

There will be 3 questions in the exams. Rectilinear is present in every of the problem in some sort of way. We will most definitely have a pulley problem. There will be a cylindrical polar problem, which may include using another system. Pulleys, C-P (with others), relative motion, are going to be the three exam questions. Generally, one of the problem is from the example done in class. Another is from the homework. The third is something we have not seen, but should be solvable. We should give numeric answers in this, and the numbers are worth 10% here.

If we are going 80 mph as A along object B that is 60 mph, object B appears to be going behind by 20 mph. The notation for the position is

$$\vec{r}_{B/A} \tag{46}$$

which is read as 'B with respect to A'.

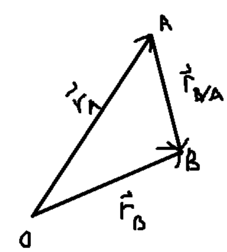

Figure 11: Relative Position

In figure 11:

$$\vec{r}_A + \vec{r}_{B/A} = \vec{r}_B \tag{47}$$
$$\vec{v}_A + \vec{v}_{B/A} = \vec{v}_B \tag{48}$$
$$\vec{a}_A + \vec{a}_{B/A} = \vec{a}_B \tag{49}$$

We will have to use Cartesians to convert coordinate systems if it was, for example, N-T.

Example 14

Two boats leave the shore at the same time and travel in the directions shown. If v A and v B are 20 ft/s and 15 ft/s, respectively, determine the velocity of boat A with respect to boat B. How long after leaving the shore will the boats be 800 ft apart?

Things we know:

$$v_A = 20\,\text{ft/s}$$
$$v_B = 15\,\text{ft/s}$$
$$\alpha = 30^\circ$$
$$\beta = 45^\circ$$
$$d = 800\,\text{ft}$$

and asked to find $\vec{v}_{A/B}$ and t_f.

$$\begin{aligned}\vec{v}_{A/B} &= \vec{v}_A - \vec{v}_B \\ \vec{v}_A &= v_A \hat{u}_{TA} \\ \vec{v}_B &= v_B \hat{u}_{TB}\end{aligned}$$

plugging this into N-T coordinates with $\hat{i}, \hat{j}$:

$$\begin{aligned}\hat{u}_{TA} &= (-\sin\alpha)\hat{i} + (\cos\alpha)\hat{j} \\ \hat{u}_{TB} &= (\cos\beta)\hat{i} + (\sin\beta)\hat{j}\end{aligned}$$

we get

$$\begin{aligned}\vec{v}_A &= v_A\left[(-\sin\alpha)\hat{i} + (\cos\beta)\hat{j}\right] \\ \vec{v}_B &= v_B\left[(\cos\beta)\hat{i} + (\sin\beta)\hat{j}\right]\end{aligned}$$

$$\vec{v}_{A/B} = (-v_A\sin\alpha - v_B\cos\beta)\hat{i} + (v_A\cos\alpha - v_B\sin$$

$$\begin{aligned}\vec{v}_{A/B} &= -20.61\hat{i} + 6.71\hat{j}\ \text{[ft/s]} \\ &= v_x\hat{i} + v_y\hat{j}\ \text{[ft/s]}\end{aligned}$$

We take the integral:

$$\int \vec{v}_{A/B}\,\mathrm{d}t = \vec{r}_{A/B}$$

$$= (v_x t + \cancelto{0}{C_1})\hat{i} + (v_y t + \cancelto{0}{C_2})\hat{j}$$

and solve for the magnitude

$$\begin{aligned}|\vec{r}_{A/B}| = \sqrt{(v_x t)^2 + (v_y t)^2} &= d \\ t &= 36.9\,\mathrm{s}\end{aligned}$$

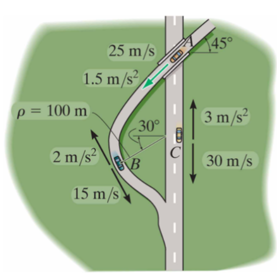

Figure 12: Example 15

Example 15

Car B is traveling along the curved road while braking. At this same instant, Car C is traveling along the straight road while also braking. Determine the velocity and acceleration of car B relative to car C.

Given:

$$\begin{aligned}v_B &= 15\,\mathrm{m/s} \\ \dot{v}_B &= -2\,\mathrm{m/s^2} \\ \theta &= 30^\circ \\ \rho &= 100\,\mathrm{m} \\ v_C &= 30\,\mathrm{m/s} \\ \dot{v}_C &= -v\,\mathrm{m/s^2}\end{aligned}$$

and asked to find $\vec{v}_{B/C}$ and $\vec{a}_{B/C}$. Each particle wants to be in N-T, but for conversion, we need Cartesian. This problem will be on the exam. Our equations for relative motion are:

$$\begin{aligned}\vec{v}_{B/C} &= \vec{v}_B - \vec{v}_C \\ \vec{a}_{B/C} &= \vec{a}_B - \vec{a}_C\end{aligned}$$

In N-Ts, we have:

$$\begin{aligned}\vec{v}_B &= v_B\hat{u}_{TB} \\ \vec{v}_C &= v_C\hat{u}_{TC} \\ \vec{a}_B &= \dot{v}_B\hat{u}_{TB} + \frac{v_B^2}{\rho}\hat{u}_{NB} \\ \vec{a}_C &= \dot{v}_C\hat{u}_{TC} + \cancelto{0}{\frac{v_C^2}{\rho}}\hat{u}_{NB}\end{aligned}$$

We convert these to Cartesians. We should not change the rotation of the $\hat{i}$ and $\hat{j}$ coordinates.

$$\begin{aligned}\hat{u}_{TB} &= (\sin\theta)\hat{i}(-\cos\theta)\hat{j} \\ \hat{u}_{NB} &= (\cos\theta)\hat{i}(\sin\theta)\hat{j} \\ \hat{u}_{TC} &= -\hat{j}\end{aligned}$$

Plugging the above coordinate system into the N-T coordinates, we have:

$$\begin{aligned}\vec{v}_B &= v_B\left[(\sin\theta)\hat{i} + (-\cos\theta)\hat{j}\right] \\ \vec{v}_C &= v_C\left[-\hat{j}\right] \\ \vec{a}_B &= \dot{v}_B\left[(\sin\theta)\hat{i} + (-\cos\theta)\hat{j}\right] + \frac{v_B^2}{\rho}\left[(\cos\theta)\hat{i} + (\sin\theta)\hat{j}\right] \\ \vec{a}_C &= \dot{v}_C\left[-\hat{j}\right]\end{aligned}$$

Plugging this in together and re-arranging:

$$\vec{v}_{B/C} = (v_B \sin\theta - 0)\,\hat{i} + (-v_B \cos\theta + v_C)\,\hat{j}$$
$$\vec{a}_{B/C} = \left(\dot{v}_B \sin\theta + \frac{v_B^2}{\rho}\cos\theta\right)\hat{i} + \left(-\dot{v}_B \cos\theta + \frac{vB}{\rho}\right.$$

Plugging in the values, we get

$$\vec{v}_{B/C} = 7.5\hat{i} + ?\hat{j} \ \text{[m/s]}$$
$$\vec{a}_{B/C} = 0.95\hat{i} - 0.14\hat{j} \ \left[\text{m/s}^2\right]$$

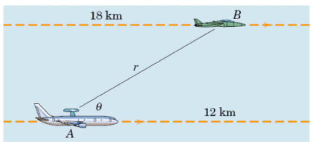

Figure 13: Example 16

Example 16

AWACS is flying at a constant altitude of 12km and is increasing it's speed at a rate of 1.2 m/s each second. It has a radar lock on aircraft B flying in the same direction at a constant altitude of 18km at a constant speed of 1500kph. If A has a speed of 1000kph at the instant where θ is 30°, determine the values for $\ddot{r}$, and $\ddot{\theta}$.

We are given

$$\dot{v}_A = 1.2\,\text{m/s}^2$$
$$H = 6000\,\text{m}$$
$$v_B = 1500 \times \frac{1000}{3600}\,\text{m/s}$$
$$v_A = 1000 \times \frac{1000}{3600}\,\text{m/s}$$

and asked to find $\ddot{r}$ and $\ddot{\theta}$. The radar sees in cylindrical polars. It does not detect the *absolute* motion. It sees the relative motion of B with respect to A. We can use Cartesian to glue their coordinates together, but then use C-P for the final answer. Equations of relative motion are:

$$\vec{v}_{B/A} = \vec{v}_B - \vec{v}_A$$
$$\vec{a}_{B/A} = \vec{a}_B - \vec{a}_A$$

The tangent direction is to the right in N-Ts, for both. $\hat{u}_N$ does not matter.

$$\vec{v}_A = v_A \hat{u}_T$$
$$\vec{v}_B = v_B \hat{u}_T$$
$$\vec{a}_A = \dot{v}_A \hat{u}_T + \cancelto{0}{\frac{v_A^2}{\rho}}\hat{u}_N$$
$$\vec{a}_B = 0$$

Technically, ρ is the curvature of the Earth plus their altitude, but it is so large that we can consider it infinite. $\vec{a}_B$ is zero because it is going at a constant rate. We now convert N-T to $\hat{i}$ and $\hat{j}$:

$$\hat{u}_T = \hat{i}$$
$$\therefore \vec{v}_{B/A} = (v_B - v_A)\hat{i}$$
$$\text{and } \vec{a}_{B/A} = -\dot{v}_A \hat{i}$$

According to the radar, which is in C-P, has its terms in radius and theta, which should also be equal to the above:

$$\vec{v}_{B/A} = \dot{r}\hat{u}_r + r\theta\hat{u}_\theta \qquad = (v_B - v_A)\hat{i}$$
$$\vec{a}_{B/A} = (\ddot{r} - r\dot{\theta}^2)\hat{u}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\hat{u}_\theta \qquad = -\dot{v}_A\hat{i}$$

and the conversion for the unit vectors are

$$\hat{u}_r = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$
$$\hat{u}_\theta = (-\sin\theta)\hat{i} + (\cos\theta)\hat{j}$$

and plugging them in to the N-T equations:

$$\dot{r}\left[(\cos\theta)\hat{i} + (\sin\theta)\hat{j}\right]\ldots$$
$$+r\theta\left[(-\sin\theta)\hat{i} + (\cos\theta)\hat{j}\right] = (v_B - v_A)\hat{i}$$
$$(\ddot{r} - r\dot{\theta}^2)\left[(\cos\theta)\hat{i} + (\sin\theta)\hat{j}\right]\ldots$$
$$+\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\left[(-\sin\theta)\hat{i} + (\cos\theta)\hat{j}\right] = -\dot{v}_A\hat{i}$$

Also, we can find r using the height and angle:

$$\frac{H}{r} = \sin\theta$$

We will have to use MATLAB to solve this.

We can use MATLAB for finding a solution by:

```
clc; % clear screen
clear; % clear memory
A = 1;
B = 2;
```

```
C = 1;
D = 4;
syms x y % unevaluated terms
eqn1 = A * x + B * y == 1
eqn2 = C * x + D * y == 2
sol = solve([eqn1, eqn2], [x, y]);
% To display:
X = double(sol.x)
Y = double(sol.y)
```

3/03: Newton's Laws

Up until now, we have only talked about how to talk about motion. In ENME Statics, we spend most the

	Particles	Rigid Bodies
Kinematics	Exam 1	
Kinetics	Current	

Table 3: Quadrants of our study

time dealing with the left hand side of the equation

$$\sum \vec{F} = m\vec{a} \tag{50}$$

but in this class, we focused on the right hand side of the equation. Dynamics problem wants to be solved by N-Ts, unless there is a good reason to use something else, particularly C-Ps.

List of steps for Newton's Second Law problems:

1. Draw Free Body Diagram (FBD)
2. Draw mass-acceleration diagram (MAD)
3. Coordinate System (N-T, unless C-P)
4. $\sum \vec{F} = m\vec{a}$
5. System summary
6. Use extra information from: friction laws, spring laws, kinematics (how it is moving) or geometry (its current configuration).
7. Solve, celebrate, or iterate

We can NOT rotate the free bodies.

Example 20

Blocks A and B (of mass 10kg and 6kg, respectively) are placed on the inclined plane and "gently nudged" from rest to slide down the hill. Determine the internal force in the link. The coefficient of kinetic friction between the blocks and the inclined plane are 0.1 and 0.3 for blocks A and B, repectively Neglect the mass of the link.

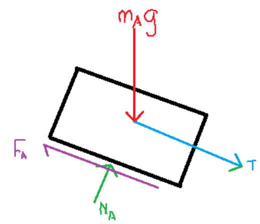

Figure 14: Example 20, FBD for A

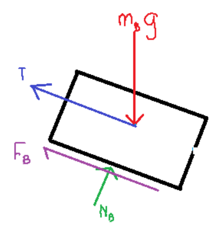

Figure 15: Example 20, FBD for B

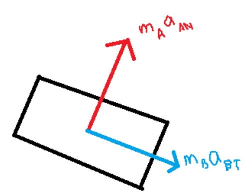

Figure 16: Example 20, MAD for A

This is a Newton's second law problem. We need two FBDs because they are pinned. If they were welded together, it would have been 1. Given:

$$m_A = 10\,\text{kg}$$
$$m_B = 6\,\text{kg}$$
$$\mu_A = 0.1$$
$$\mu_S = 0$$
$$\theta = 30^\circ$$

Gravity is present. In Statics, we assumed trusses to be in either pure tension or compression. So long as there is no force on the truss (including its mass), and it is pinned at the ends, then it can be considered a two-force member. Friction will always try to not make us move at all in the first place. Secondly, it works in the opposite direction of wherever we are trying to go. This problem wants to be solved in N-Ts. Write down the sum of the forces.

$$T - F_A + m_A g \sin\theta = m_A a_{AT}$$
$$N_A - m_A g \cos\theta = m_A a_{AN}$$
$$-T - F_B + m_B g \sin\theta = m_B a_{BT}$$
$$N_B - m_B g \cos\theta = m_B a_{BN}$$

System summary means to count the equations, and list the unknowns. In the above, we have four equations, but nine unknowns. When we go really fast, our friction decreases because there has an air underneath it. It is called lubrication theory. We know from friction law that:

$$F_A = \mu_K N_A$$
$$F_B = \mu_K N_B$$

Due to infinite curvature in the N-T coordinate,

$$a_{AN} = 0$$
$$a_{BN} = 0$$
$$\rho = \infty$$

We will pretend that the bar is rigid, though the bar would compress and decompress in real life. Since they are moving together,

$$a_{AT} = a_{BT}$$

Now we have 9 equations, 9 unknowns. In an exam, it would be sufficient to just have the system of equations be our answers; we should write down the units of all the variables we are expecting. Also, box the variables and the value we are *expecting*. Plugging the values in MATLAB, we get:

$$T = -6.37\,\text{N}$$
$$F_A = 8.49\,\text{N}$$
$$F_B = 15.29\,\text{N}$$
$$a_{AT} = 3.41\,\text{m/s}^2$$
$$a_{AN} = 0$$
$$a_{BT} = 3.41\,\text{m/s}^2$$
$$a_{BN} = 0$$
$$N_A = 84.9\,\text{N}$$
$$N_B = 50.97\,\text{N}$$

This methodology works anytime, and may be applicable in other classes as well. Also, we do not necessarily have to do the trigonometry in the professor's way.

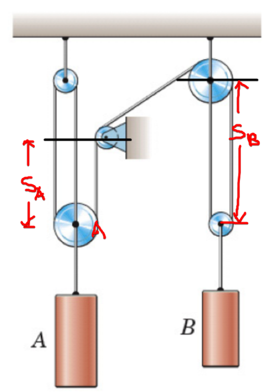

Figure 17: Example 21

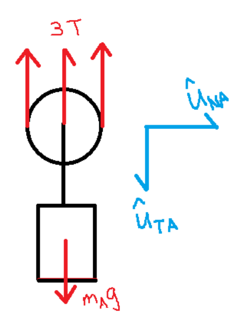

Figure 18: Example 21 FBD A

Example 21

Masses A and B are of 10kg and 7kg, respectively. Determine the accelerations of the masses

We should deal with the 'pulliness' first (the length being consistent)

$$L = 3s_A + 2s_{B-1} + (\dots)$$
$$0 = 3a_{AT} + 2a_{BT}$$

Since we are ignoring the pulleys for this semester, we can draw a FBD that bites into the working cable. By biting into, we are meaning that we are showing the forces on something. Given:

$$m_A = 10\,\text{kg}$$
$$m_B = 7\,\text{kg}$$

We can assume the cable tension to be the same, because we assume mass of the pulleys to be zero. Our tangent direction is down for A, and the N is either left or right. Our equations are

$$m_A g - 3T = m_A a_{AT}$$
$$m_B g - 2T = m_B a_{BT}$$

and I have three unknowns, therefore the third equation is

$$3a_{AT} + 2a_{BT} = 0.$$

Using MATLAB,

$$a_{AT} = -0.19\,\text{m/s}^2$$
$$a_{BT} = 0.286\,\text{m/s}^2$$
$$T = 33.3\,\text{N}$$

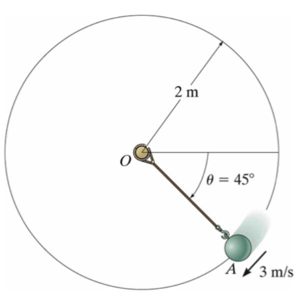

Figure 19: Example 22

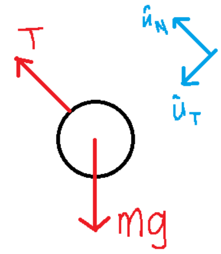

Figure 20: Example 22 FBD

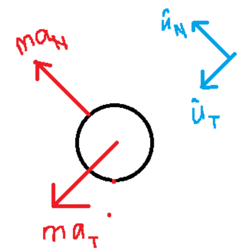

Figure 21: Example 22 MAD

Example 22

If the 10kg ball has a velocity of 3 m/s when it is in the position A along the vertical path, determine the tension in the cord and the increase in the speed of the ball at this position.

We are given

$$\theta = 45°$$
$$v = 3\,\text{m/s}$$
$$m = 10\,\text{kg}$$
$$L = 2\,\text{m}$$

We will have to use N-Ts for this, and are asked to find T, and the rate of speed increase, at this very instant. Although this is circular motion, we use N-Ts because the motion is in terms of the ball. When $\theta \to 0$, all of gravity's force points in the tangent direction. Therefore:

$$mg\cos\theta = ma_T$$
$$T - mg\sin\theta = ma_N$$

and an extra equation

$$a_N = \frac{v^2}{\rho} = \frac{v^2}{L}.$$

This lets the three unknowns be

$$
\begin{aligned}
T &= 114\,\mathrm{N} \\
a_T &= 6.94\,\mathrm{m/s^2} \\
a_N &= 4.5\,\mathrm{m/s^2}
\end{aligned}
$$

Note that $a_T = \dot{v}$.

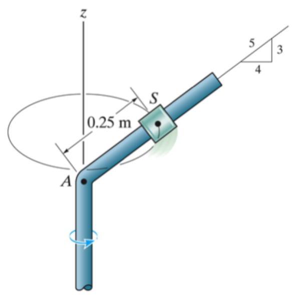

Figure 22: Example 23

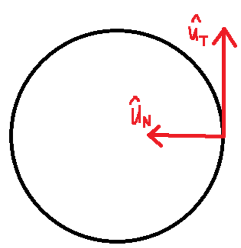

Figure 23: Example 23 Coordinates

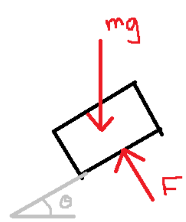

Figure 24: Example 23 FBD

Example 23

The 2kg spool S fits loosely on the smooth inclined rod. Determine the constant speed the spool must have such that it will not move.

There is no friction because it smooth. We are given

$$
\begin{aligned}
m &= 2\,\mathrm{kg} \\
\theta &= \arctan \frac{3}{4} \\
L &= 0.25\,\mathrm{m}
\end{aligned}
$$

If we spin this too fast, the spool will fly out, while too slow will make it fall towards A. There is no centripetal force, but rather, contact force. We have our equations

$$
\begin{aligned}
F \sin\theta &= m a_N \\
-mg + F\cos\theta &= m a_B
\end{aligned}
$$

but we have four equations: a_N, a_B, F, v, ρ. We are asked to find velocity but that is not in our equations, but we should still include it in the unknowns. We know that we have no friction. Note that a_B is the acceleration in the binormal direction. We know that it doesn't move up and down, and we have a equation for the normal component of acceleration that uses ρ:

$$
\begin{aligned}
a_B &= 0 \\
a_N &= \frac{v^2}{\rho} \\
\rho &= L\cos\theta
\end{aligned}
$$

Plugging these in MATLAB, we can find the solution.

3/10: More force problems

The HW will be due on Wednesday.

In example 24, the sign convention according to the cylindrical polar is outside. Since friction acts in opposite direction, friction is negative. Since the object in this problem does not slide, the change in z is zero. There will be an acceleration inwards.

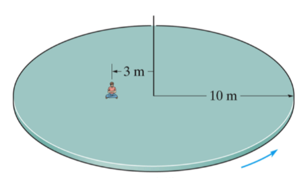

Figure 25: Example 26

Example 26

The man has a mass of 80kg and sits 3m from the center of the rotating platform. Due to rotation his speed is increased by 0.4m/s2 . If the coefficient of static friction between his clothes and the platform is 0.3, determine the time required to cause him to slip.

We are looking for t. The information we are given is in cylindrical polars, so we will use C-P coordinates. Unless there is something very definitely C-P, we should use N-Ts. Our knowns are

$$\begin{aligned} \dot{v} &= a_T = 0.4\,\mathrm{m/s^2} \\ \mu_s &= 0.3 \\ L &= 3\,\mathrm{m} \end{aligned}$$

The person would keep spinning until the friction can no longer can keep him there. At the instant of flying off, they will fly off in the $\hat{u}_T$ direction. Gravity is guaranteed acting down, but all the other forces (asides normal force) are acting in other directions. Therefore, we have to use a top view.

There has to be a friction that allows them to rotate, but also not fly off. Therefore, the friction force has to be partly in the tangent direction, and inward.

If we were going contact speed, $\dot{v}$ would be zero, and the friction would only be towards the center.

$$\begin{aligned} B: &\quad F\cos\alpha = ma_T \\ T: &\quad F\sin\alpha = ma_N \\ Z: &\quad N - mg = 0 \end{aligned}$$

We are concerned about the top view, so we do not even need to consider the z axis. We also know that

$$\begin{aligned} a_T &= \dot{v} \\ F &= \mu_s N \\ a_N &= \frac{v_f^2}{L} \\ v_f &= \dot{v} t_f \end{aligned}$$

If we can not solve the equation due to algebraic complexity (but we have our equations right), then we can just write the units of each of the missing variables and get most of the points. The α can be radians or degrees.

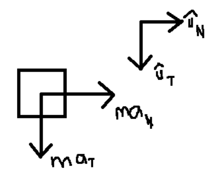

Figure 26: Example 26 Directions

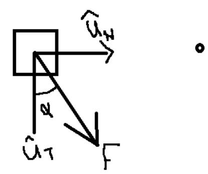

Figure 27: Example 26 Friction Force

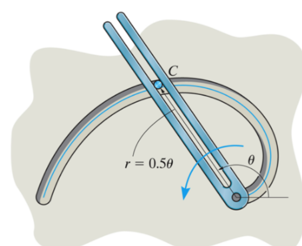

Figure 28: Example 25

Example 25

There is no mass in this problem. We are given:

$$\begin{aligned} m &= 0.5\,\mathrm{kg} \\ \theta &= 0.5t^2\,\mathrm{m} \\ r &= 0.5\theta\,\mathrm{m} \\ &= 0.25t^2\,\mathrm{m} \\ t_f &= 2\,\mathrm{s} \end{aligned}$$

The cylinder has to be in contact with only side of the forked rod because it has to be able to slide. Draw the MAD and FBD diagrams. We don't know the angle that the slot makes with the unit coordinates. Our two equations are:

$$\begin{aligned} r: \quad & -N_s \cos\phi = ma_r \\ \theta \quad & N_s \sin\phi + N_B = ma_\theta \end{aligned}$$

and additionally, we can write

$$\begin{aligned} a_r &= \ddot{r}_f - r_f \dot{\theta}_f^2 \\ a_\theta &= r_f \ddot{\theta}_f + 2\dot{r}_f \dot{\theta}_f \\ r_f &= 0.25 t_f^2 \\ \dot{r}_f &= 0.5 t_f \\ \ddot{r}_f &= 0.5 \\ \theta &= 0.5t^2 \\ \dot{\theta} &= t \\ \ddot{\theta} &= 1 \end{aligned}$$

The last equation needed to solve the problem is

$$\vec{v} = \dot{r}_f \hat{u}_r + r_f \dot{\theta}_f \hat{u}_\theta$$

which is in the same direction (left) as the reference line in the FBD in 29.

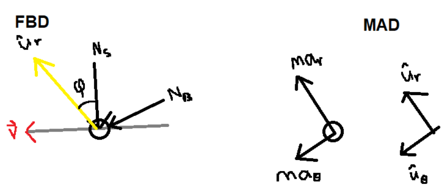

Figure 29: Example 25 MAD and FBD

There is no quiz on Monday. Exams should be graded in a week.

3/15: Work Energy Conservations

Today we will learn about what is learned in the beginning of a vibrations course. We will learn the work-energy method. This week will be easy for us.

The first law of energy cannot be created or destroyed. The energy is transferred. Drawing a dotted line in a figure, a control volume, the energy in it will be conserved.

Everything on the right-hand side of equation 51 is those that can be regained (kinetic, potential). In a garage door, the spring has potential energy (this is denoted by ΔE_s, where s could stand for spring, or strain).

$$W = \Delta E_K + \Delta E_P + \Delta E_s \tag{51}$$

Mathematically, work is defined by an integral of dot product, but many cases, we can simplify it as

$$W = \int \vec{F} \cdot d\vec{r} = F_\perp d \tag{52}$$

The work done by friction is negative, because the dot product term yields a negative. The negative can be interpreted as leaving the system. Friction is lazy, contrarian, and sucks energy out of the system. Referring to 30, the force from gravity and normal

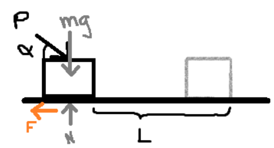

Figure 30: Example forces

force has no work done because it is perpendicular to direction of travel.

A spring has a constant k, and a free length when no applied force present. When a force F_s is applied, the spring compresses/expands by s. F_s and s are linearly independent only while it is in the liner elastic region. In our class, we will consider a positive F_s to be stretching. We will use

$$F_s = ks \tag{53}$$

In our class

$$s = L_s - L_{free} \tag{54}$$

The equations for the other forms of energy are

$$\Delta E_K = \frac{1}{2} m \left(v_f^2 - v_0^2\right) \tag{55}$$

$$\Delta E_p = mg\Delta y \tag{56}$$

$$\Delta E_s = \frac{1}{2} k (s_f^2 - s_0^2) \tag{57}$$

This is same as Thermodynamic's $Q - W = \Delta U$, but $Q = 0$, and we do not always assume that energy is leaving the system.

Newton's Second Law is used at a single instant. Work-energy is used over a displacement.

The end point of these example problems will be the start of our vibrations course.

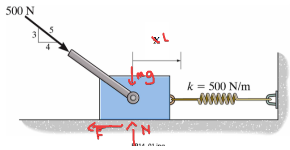

Figure 31: Example 27

Example 27

The spring is placed between the wall and the 10kg mass. If the block is subjected to a force of 500N, determine its velocity when it has moved 0.2m. The coefficient of kinetic friction is 0.1.

We are given

$$\begin{aligned} m &= 10\,\text{kg} \\ K &= 500\,\text{N/m} \\ P &= 500\,\text{N} \\ L &= 0.2\,\text{m} \\ \mu_k &= 0.1 \end{aligned}$$

and asked to find v_f. Our equation for energy is:

$$\begin{aligned} W &= \Delta E_K + \cancelto{0}{\Delta E_p} + \Delta E_s \\ W &= \frac{4P}{5} \cdot L - FL \\ \Delta E_K &= \frac{1}{2} m \left(v_f^2 - \cancelto{0}{v_0^2} \right) \\ \Delta E_s &= \frac{1}{2} k \left(s_f^2 - \cancelto{0}{s_0^2} \right) \end{aligned}$$

The work done is obtained by the components of the applied force and friction. Since the spring is attached to a wall, it implies that initial velocity is none. From an energy point of view, s_f is positive, but from force point of view, it will be $s_f = -L$. Combining the equations:

$$\begin{aligned} \left(\frac{4P}{5} - F \right) L &= \frac{1}{2} m v_f^2 + \frac{1}{2} k L^2 \\ F &= \mu_k N \\ N - mg - P\left(\frac{3}{5}\right) &= m\cancelto{0}{a_N} \end{aligned}$$

The $a_N = 0$ because radius of curvature is infinite. Solving the three equations above will yield F, v_f, N, but note that mathematically v_f will be $\pm$ because we will take the square root to solve it. In this case, we *know* that it will be positive since it is moving to the right.

The take home quiz that we will have a spring in it.

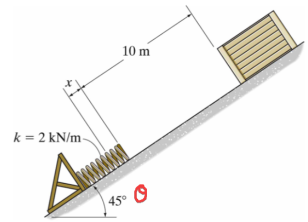

Figure 32: Example 28

Example 28

The coefficient of kinetic friction between the initially stationary 100kg crate and the hill is 0.25. Determine the max compression of the spring.

This problem is work energy because of the large displacement. We are given

$$\begin{aligned} m &= 100\,\text{kg} \\ \mu_k &= 0.25 \\ L &= 10\,\text{m} \\ v_0 &= 0\,\text{m/s} \\ \theta &= 45° \end{aligned}$$

and are asked two find x. We will assume that the spring is max compressed, therefore the final velocity is zero:

$$\begin{aligned} W &= \cancelto{0}{\Delta E_K} + \Delta E_P + \Delta E_S \\ W &= -F(L + x) \\ \Delta E_P &= -mg(L + x)\sin\theta \\ \Delta E_S &= \frac{1}{2} k \left(s_f^2 - \cancelto{0}{s_0^2} \right) \end{aligned}$$

Because we are going down, potential energy is negative. Since there is friction, work done is also negative. Since $s_f = -x$, plugging the equations in we get

$$-F(L+x) = -mg(L+x)\sin\theta - \frac{1}{2}kx^2$$
$$F = \mu_k N$$
$$N - mg\cos\theta = m\cancelto{0}{a_N}$$

the a_n term is zero because $\rho \to \infty$.

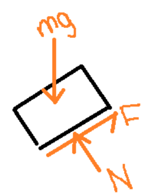

Figure 33: Example 28 FBD

In a sophomore dynamics class, we don't have to worry about the mass of the spring. In later courses we will.

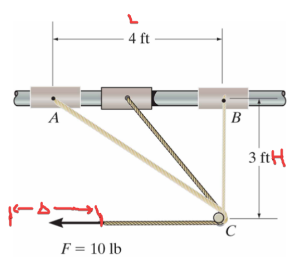

Figure 34: Example 29

Example 29

The 5lb collar is pulled by a cord that passes around a small peg at C. If the cord is subjected to a constant force of 10lb and the collar is at rest at A, determine its speed when it reaches B. Neglect friction and the weight of the cord.

We are given

$$m = \frac{5}{32.2}\,\text{slug}$$
$$F = 10\,\text{lb}$$
$$L = 4\,\text{ft}$$
$$H = 3\,\text{ft}$$
$$v_A = v_0 = 0\,\text{ft/s}$$

and asked to find $v_B = v_f$. There is no friction, there are tension. Also, note that mass unit in imperial is slug. There is no change in height, nor a spring, so the equations are:

$$W = \Delta E_K + \cancelto{0}{\Delta E_p} + \cancelto{0}{\Delta E_s}$$
$$W = F\Delta$$
$$\Delta E_K = \frac{1}{2}m\left(v_f^2 + \cancelto{0}{v_0^2}\right)$$
$$\Delta = \sqrt{H^2+L^2} - H$$

Cable AC starts as 5 feet, and BC is 3 feet. Since the total cable length has to be constant, the $\Delta = 2\,\text{ft}$. Plugging these together:

$$F\Delta = \frac{1}{2}mv_f^2$$

The velocity final will be mathematically positive and negative, but we have to choose which sign it will be.

3/17: More Work Problems

In this class, 'smooth' means frictionless.

Example 30

The vertical guide is smooth and the 5kg collar is released from rest at A. Determine the speed of the collar when it is at position C. The spring has an unstretched length of 300mm.

There is kinetic, potential, and spring energy here, but there is no work done. Potential energy is negative because final height is lower than the initial.

$$\cancelto{0}{W} = \Delta E_K + \Delta E_P + \Delta E_S$$

$$\Delta E_K = \frac{1}{2}m(v_f^2 - \cancelto{0}{v_0^2})$$

$$\Delta E_P = -mgH$$

$$\Delta E_S = \frac{1}{2}k(s_f^2 - s_0^2)$$

The original length $s_0 = L - L_{free}$, while the final is $s_f = \sqrt{H^2 + L^2} - L_{free}$. We should place the final equation in one line:

$$0 = \frac{1}{2}mv_f^2 - mg\pi + \frac{1}{2}k((\sqrt{H^2 + L^2} - L_{free})^2 - (L - L_f$$

If you get an imaginary number when doing the math, that implies that either the math was done wrong, or the particular action does not ever happen.

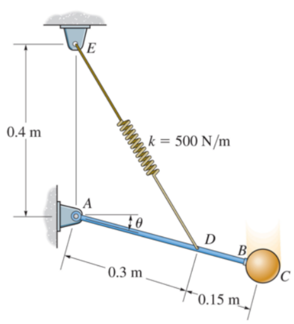

Figure 35: Example 31

Example 31

The 10kg sphere C is released from rest when q=0° and the tension in the spring is 100N. Determine the speed of the sphere at the instant q=90°. Neglect the mass of the rod and the size of the sphere (at least for the next few weeks...)

We will assume that the spring will not hit the hinge. Given:

$$H = 0.4\,\text{m}$$
$$L_1 = 0.3\,\text{m}$$
$$L_2 = 0.15\,\text{m}$$
$$m = 10\,\text{kg}$$
$$k = 500\,\text{N/m}$$
$$T_0 = 100\,\text{N}$$

We are given the initial tension. We know this is a work displacement problem because it has a large displacement. There is no net work done. The other forms of energy changes are present:

$$\cancelto{0}{W} = \Delta E_K + \Delta E_P + \Delta E_S$$

$$\Delta E_K = \frac{1}{2}m(v_f^2 - \cancelto{0}{v_0^2})$$

$$\Delta E_P = -mg(L_1 + L_2)$$

$$\Delta E_S = \frac{1}{2}k(s_f^2 - s_0^2)$$

We can use the initial tension in the spring to determine s_0, and also L_{free}:

$$T_0 = ks_0 \implies \frac{T_0}{k} = s_0 = L_0 - L_{free}$$

The final combined equation is

$$0 = \frac{1}{2}mv_f^2 - mg(L_1 + L_2)\cdots$$
$$+\frac{1}{2}k\left(\left[H + L_1 - \left(\sqrt{H^2 + L_1^2} - \frac{T_0}{K}\right)\right]^2 - \left(\frac{T_0}{k}\right)^2\right)$$

Example 32

The 2kg pendulum bob is released from rest when it is at A. Determine the speed of the mass and the tension in the cable when the mass passes it's lowest point at B

We are given

$$m = 2\,\text{kg}$$
$$L = 1.5\,\text{m}$$

and are asked to find the v_f and tension in the cable. Here, there are two different dynamic problems: swinging which is a work energy problem, and Newton's Second Law. For the second state, tension does no work because it is perpendicular in B, therefore we have to use Newton's Laws. Part A: The work done equation is:

$$\cancelto{0}{W} = \Delta E_k + \Delta E_p + \cancelto{0}{\Delta E_s}$$
$$\Delta E_k = \frac{1}{2}m(v_f^2 - \cancelto{0}{v_0^2})$$
$$\Delta E_p = -mgL$$

Tension is always perpendicular to the path, so it does no work. Although our current examples have 0s for many of the terms, we should not assume them to be at all times. The equation has only one unknown that allows us to solve for $v_f = 5.43\,\text{m/s}$:

$$0 = \frac{1}{2}mv_f^2 - mgL$$

Part B: For the tension, however, we need to use a free body diagram. We do not have to draw FBDs for work problems because it is scalar, but this is not. Figure 36 shows the FBD and MAD. The equations in this are

$$T - mg = ma_N$$
$$a_N = \frac{v_f^2}{L}$$

At the vertical position, there is no horizontal acceleration at that very instant.

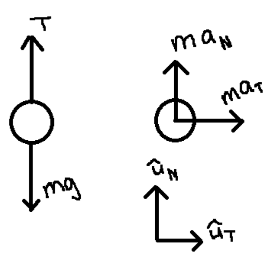

Figure 36: Example 32 FBD and MAD

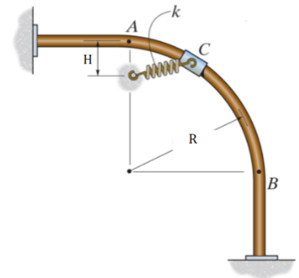

Figure 37: Example 33

Example 33

A smooth collar of given mass (m) is pin connected to a spring of given stiffness (k) with a known free length (L free). The mass is initially held at point B then released from rest. The plane is vertical.

1. Obtain a single equation usable to determine the speed of the collar at point A.
2. Obtain a closed set of equations usable to determine the force of the track on the collar immediately to the right of point A (on the quarter circular segment).
3. Obtain a closed set of equations usable to determine the force of the track on the collar immediately to the left of point A (on the straight segment).

The difference between part 2 and 3 is that in part 2, the radius of curvature is R, while in part 3 it is infinite. For part 1, we will use work energy. While for the other two, we will use Newton's Second Law.
We are given m, R, H, K, L_{free}. The work energy equation is

$$\cancelto{0}{W} = \Delta E_k + \Delta E_p + \Delta E_s$$
$$\Delta E_K = \frac{1}{2}m(v_f^2 - v_0^2)$$
$$\Delta E_P = mgR$$
$$\Delta E_S = \frac{1}{2}k(s_f^2 - s_0^2)$$

Note that we are keeping the v_0 term because the professor wants to illustrate what happens to the term. We can find s_0 using Pythagoras

Theorem:

$$s_0 = \sqrt{(R-H)^2 + R^2} - L_{free}$$
$$s_f = H - L_{free}$$

note that s_f can be negative. The final equation is

$$0 = \frac{1}{2}m(v_f^2 - v_0^2) - mgR \ldots$$
$$+ \frac{1}{2}k\left[(H - L_{free})^2 - \left(\sqrt{(R-H)^2 + R^2}\right)\right]^2$$

to get v_f. For part 2 and 3, assume tension. If it is compressed, the math will just say negative. Figure 38 shows the FBD and MAD for part 2.

$$0 = ma_T$$
$$mg + F_s - N = ma_N$$
$$F_s = ks_f$$

The unknowns of the equation are a_T, a_N, F_s, N, but a_T is 0 by default at that very instant. The difference between part 2 and 3 is that in 2,

$$a_N = \frac{v_f^2}{R}$$

while in part 3:

$$a_N = 0$$

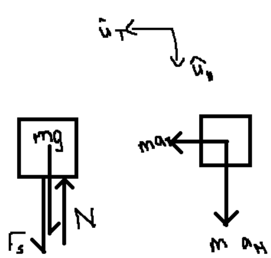

Figure 38: Example 33 Diagrams for 1

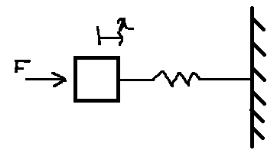

Figure 39: Professor's Example

In the self example problem by the professor:

$$W = \Delta E_K + \cancel{\Delta E_P}^{0} + \Delta E_S$$
$$W = \int F \cdot \mathrm{d}x$$
$$= Fx$$
$$\Delta E_K = \frac{1}{2}m(v_f^2 - \cancel{v_0^2}^{0})$$
$$\Delta E_S = \frac{1}{2}k(s_f^2 - \cancel{s_0^2}^{0})$$
$$\Rightarrow Fx = \frac{1}{2}mv_f^2 + \frac{1}{2}kx^2$$

We can remove v_f since it is $\dot{x}$. Also, we can take the derivative on both sides:

$$F\dot{x} = m\dot{x}\ddot{x} + kx\dot{x} \quad (58)$$
$$\Rightarrow 0 = (m\ddot{x} + kx - F)\,\dot{x} \quad (59)$$
$$F = m\ddot{x} + kx \quad (60)$$

The last equation 60, is the second order equation of motion for vibrations.

Homework will be due the Wednesday after spring break. The Monday after spring break we will have a quiz on Newton's Second Law. Additionally, we should not add or subtract from the population.

3/29: Linear Impulse for Particles

Our second exam is next week Friday. If there is a spring, it doesn't necessarily mean work-energy methods. We are learning linear impulse because of short duration effects. Newton's Second Law, Work Energy, and Impulse momentum will be on the exam.

If work energy is integration with space, impulse is with respect to time.

$$\Sigma\vec{F} = m\vec{a} \tag{61}$$

$$\int_{t_1}^{t_2} m\vec{a} = m\vec{v}\Big|_{t_1}^{t_2} \tag{62}$$

$$\Rightarrow m\vec{v}_1 + \int_{t_1}^{t_2} \Sigma\vec{F}\,\mathrm{d}t = m\vec{v}_2 \tag{63}$$

The integral in the final equation above is the impulse. Integrating force with time gives force times time, but with space gives us energy.

Graphically, impulse is the area under a force time graph. We could determine the average force by dividing the impulse with the change in time.

There are sensors on a car that eject the airbags. For any pedestrian hit by the front during the impact, they are hit by a huge force in a short period of time when their head hits the engine.

For a particle with multiple forces affecting on it, integrate each force with time, and then sum it up, to get the sum impulse and change in velocity vector. We will have to use Cartesian for this.

We can add forces up and then take the impulse when using Cartesian.

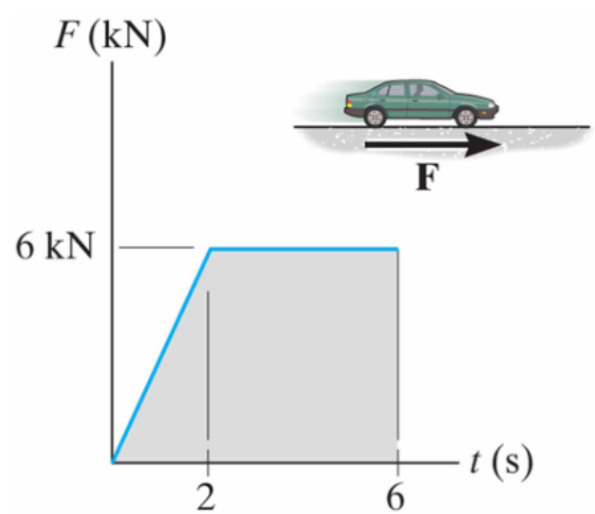

Figure 40: Example 34

Example 34

The wheels of the 1.5Mg car generate the traction force F described by the graph. If the car starts from rest, determine its speed after 6 seconds.

We are given

$$m = 1500\,\mathrm{kg}$$
$$v_0 = 0\,\mathrm{m/s}$$
$$t_f = 6\,\mathrm{s}$$
$$t_0 = 0\,\mathrm{s}$$

and asked to find v_f. We integrate the x and y components:

$$0 + \int_0^{t_f} F\,\mathrm{d}t = mv_f$$

$$0 + \int_0^{t_f} (N - mg)\,\mathrm{d}t = 0$$

The latter yields zero. The x component yields

$$v_f = \frac{1}{m}\int_0^{t_f} F\,\mathrm{d}t \quad = \frac{1}{1500\,\mathrm{kg}}(0.5 \cdot 2\,\mathrm{s} \cdot 6000\,\mathrm{N} + 4\,\mathrm{s} \cdot 6000\,\mathrm{N})$$
$$= 20\,\mathrm{m/s}$$

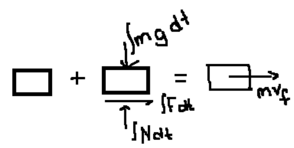

Figure 41: Example 34 IMD

When driving a car, it is the friction on the tires that actually drive us. If we do not know what component (e.g. x or y) to have on our equation, just write all.

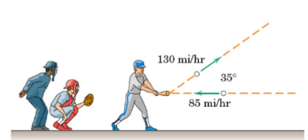

Figure 42: Example 35

Example 35

The baseball is traveling with a horizontal velocity of 85mph just before impact with the bat. Just after the impact, the velocity of the 5.125 oz ball is 130mph directed at 35° from horizontal as shown. Determine the x and y components of the average force exerted by the bat on the ball during the 0.005s impact.

We have to convert the units to smaller units:

$$
\begin{aligned}
v_0 &= 85 \cdot \frac{5280}{3600}\,\text{ft/s} \\
v_f &= 130 \cdot \frac{5280}{3600}\,\text{ft/s} \\
m &= \frac{5.125}{16} \cdot \frac{1}{32.2}\,\text{slugs} \\
\theta &= 35^\circ \\
\Delta t &= 0.005\,\text{s}
\end{aligned}
$$

We are asked to find the x and y forces. We cannot use work energy in an impulse problem. We will have to draw an impulse momentum diagram (IMD). Using it, our equations are:

$$
\begin{aligned}
-mv_0 + \int_0^{\Delta t} F_x \,\mathrm{d}t &= mv_f \cos\theta \\
0 + \int_0^{\Delta t} F_y \,\mathrm{d}t - \int_0^{\Delta t} mg \,\mathrm{d}t &= mv_f \sin\theta
\end{aligned}
$$

We can replace the integral with the average force multiplied with the time. So our x and y equations now become:

$$
\begin{aligned}
-mv_0 + \bar{F}_x \Delta t &= mv_f \cos\theta \\
\bar{F}_y \Delta t - mg\Delta t &= mv_f \sin\theta
\end{aligned}
$$

and solving for this yields

$$
\begin{aligned}
\bar{F}_y &= 217.6\,\text{lb} \\
\bar{F}_x &= 588.8\,\text{lb}
\end{aligned}
$$

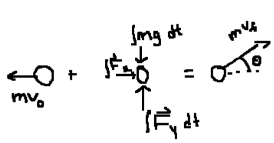

Figure 43: Example 35 IMD

During impact, we can ignore gravity because it has a very small effect. In the previous example, for example, the sum force is over 600 pounds.

It is not just the angle that affects a pool ball, but also the 'spin', which is caused by more friction on one side. However, for our case, we will ignore the friction force because we will not have the information to solve it.

So, the two things that we are ignoring are:

1. gravity
2. friction

There is generally a class on higher level electives, or in the masters level, where we finally get to see things like friction and gravity are used.

3/31: Elastic Inelastic Collisions

We will have in person quiz on Monday. Since the HW was not released, it will be due later.

Impulse-momentum is integrating force with time. It is useful for solving impact or explosive problems.

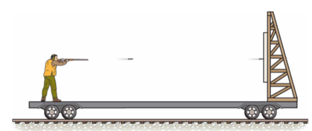

Figure 44: Example 37

Example 37

If the 150lb man fires the 0.2lb bullet with a horizontal muzzle velocity of 3000ft/s while standing on a 600lb cart, determine the velocity of the cart just after firing. What is the velocity of the cart after the bullet becomes embedded in the target?

We will assume that the person does not slide. This problem is just like if someone shot something while on roller skates. We have to convert pounds to slugs for mass.

$$
\begin{aligned}
m_A &= \frac{150 + 600}{32.2}\,\text{slug} \\
m_B &= \frac{0.2}{32.2}\,\text{slug} \\
v &= 3000\,\text{ft/s}
\end{aligned}
$$

There is no force *equation* for this, so we use impulse-momentum diagram. We need to draw two IMDs, one for the bullet, and one for the cart-man system. The shot takes really little time; the effect of gravity is very little/negligible. We get two equations:

$$0 - \int_0^{\Delta t} N\,\mathrm{d}t = m_A v_{AF}$$
$$0 + \int_0^{\Delta t} N\,\mathrm{d}t = m_B v_{BF}$$

We can use Newton's third law (notice that the integrals are equal). This way, we end up with one equation, and two unknowns:

$$0 = m_A v_{AF} + m_B v_{BF}$$

We can soon realize that we are given the velocity of the bullet *with respect* to the person, which gives us another equation:

$$v_{B/A} = v_{BF} - v_{AF}$$

solving the two equations yields

$$v_{AF} = -0.8\,\mathrm{ft/s}$$
$$v_{BF} = 2999.2\,\mathrm{ft/s}$$

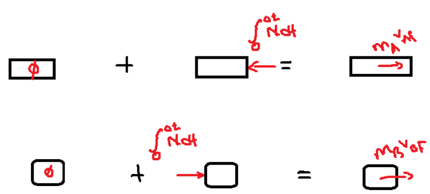

Figure 45: Example 37 IMD

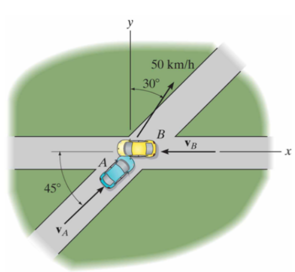

Figure 46: Example 38

Example 38

Two cars A and B have a mass of 2Mg and 1.5Mg, respectively. Determine the magnitudes of VA and VB if the cars collide and stick together while moving with a common speed of 50kph in the direction shown.

We are given

$$m_A = 2000\,\mathrm{kg}$$
$$m_B = 1500\,\mathrm{kg}$$
$$v_f = 50 \cdot \frac{1000}{3600}\,\mathrm{m/s}$$
$$\alpha = 45^\circ$$
$$\theta = 30^\circ$$

Realistically the metal would absorb the impact energy. More car crashes don't stick together. We treat this as an impulse momentum problem using two IMD diagrams. We are asked to find v_{A0}, v_{B0}.
We are recommended to *not* add masses like we were taught in our Physics class, because if the masses stick to each other, the equation will do it automatically, otherwise equation would not. Our equations for A are:

$$m_A v_A \cos\alpha - \int_0^{\Delta t} F_x\,\mathrm{d}t = m_A v_F \sin\theta$$
$$m_A v_A \sin\alpha - \int_0^{\Delta t} F_y\,\mathrm{d}t = m_A v_F \cos\theta$$

and for B are:

$$-m_B v_B + \int_0^{\Delta t} F_x\,\mathrm{d}t = m_B v_F \sin\theta$$
$$0 + \int_0^{\Delta t} F_y\,\mathrm{d}t = m_B v_F \cos\theta$$

We use Newton's third law (add the xs and the ys) to remove two equations, but remove 3 unknowns:

$$m_A v_A \cos\alpha - m_B v_B = m_A v_F \sin\theta + m_B v_F \sin\theta$$
$$m_A v_A \sin\alpha = m_A v_F \cos\theta + m_B v_F \cos\theta$$

solving yields

$$v_A = 29.77\,\mathrm{m/s}$$
$$v_B = 11.86\,\mathrm{m/s}$$

Realistically in car crashes, we do not have v_f, but we have length of the skid marks, which can be used to predict the final speed.

The other extreme of the impulse-momentum is

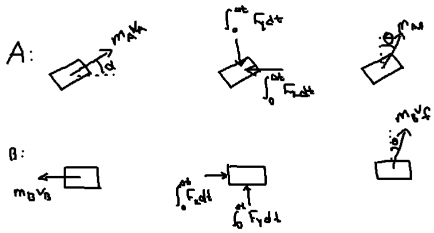

Figure 47: Example 38 IMD

elastic impact, where no energy is lost. Impacts can be inelastic and things can stick together, max energy is lost. Elastic is like flubber. Realistically, most impacts are in between.

In elastic collisions, the objects hit, deform, and then undeform as it flies away. In many materials, loading and unloading loses energy, even for elastic materials.

There is a plane of impact which is perpendicular to the surfaces that the objects hit. Parallel to this would be friction, but we are not considering friction (even though it has a major effect). We are also ignoring gravity.

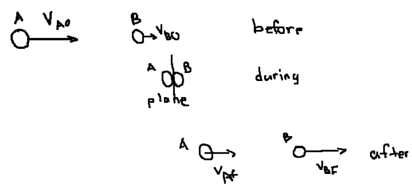

Figure 48: Elastic Collision Velocities Example

There is an extreme case of elastic impact where the velocities of 48 switch. But realistically it does not happen because even elastic materials absorb some energy. The equation for impact is

$$m_A v_{A0} + m_B v_{B0} = m_A v_{AF} + m_B v_{BF} \tag{64}$$

The coefficient of friction sets the upper limit of friction force. Friction force keeps increasing to keep an object stationary, $F_s = \mu_s N$. Then it switches to kinetic friction coefficient.

The coefficient of restitution, e, is just like that. It is given by

$$e = \frac{v_{BF} - v_{AF}}{v_{A0} - v_{B0}}, \text{where } e \in [0, 1] \tag{65}$$

or in other words, it is the 'after' over 'before', but 'switch the order'. The above equation should only involve the velocity that is *perpendicular* to the plane of impact. Only if $e = 1$ will the velocities in the beginning and final be the same, that the relative velocities would be the same (only roles switched) (upper bound of no kinetic energy lost). $e = 0$ is the extreme inelastic case (all energy is lost).

Really professional pool balls have $e \approx 0.85$. Bumpers are meant to absorb energy, hence it is lower $e \approx 0.5$. Human head is $e \approx 0.3$. A helmet has $e \approx 0.5$, but it absorbs the energy first before our head. e can never be negative. In spring mass systems, a critically damped system is where $e \approx 0$.

e is independent of the masses, just like the coefficient of friction. The velocity *parallel* to the impact plane does not change.

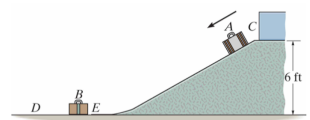

Figure 49: Example 39

Example 39

The 15lb suitcase A is released from rest at C. After is slides down the smooth ramp, it strikes the 10lb suitcase B which is originally at rest. The coefficient of restitution between the suitcases is 0.3 and coefficient of kinetic friction between the floor and each suitcase is 0.4. Determine the velocity of A before impact, the velocity of B after impact, and the distance B slides before coming to rest.

Since it is smooth, friction is negligible for the ramp. This problem has sub-problems. Work energy because of displacement of A speeding down. The impact is impulse-momentum. And later, distance traveled by B is work-energy problem. We are asked to find v_{A1}, v_{B2}, d. Our equation for (a) (refer to table 4):

$$\cancelto{0}{W} = \Delta E_k + \Delta E_p + \cancelto{0}{\Delta E_s}$$

$$0 = \frac{1}{2} m_A (v_{A1}^2 - \cancelto{0}{v_{A0}^2}) + m_A g(-H)$$

which we can use to solve $v_{A1} =$ ft/s. Looking at figure 50, we have

$$\begin{aligned} m_A v_{A1} - \int_0^{\Delta t} N \, \mathrm{d}t &= m_A v_{A2} \\ 0 + \int_0^{\Delta t} N \, \mathrm{d}t &= m_B v_{B2} \\ \Rightarrow m_A v_{A1} &= m_A v_2 + m_B v_{B2} \end{aligned}$$

The last equation has two unknowns, but the coefficient of restitution gives us the additional information:

$$e = \frac{v_{B2} - v_{A2}}{v_{A1} - v_{B1}}$$

which can be used to find v_{B2} in ft/s. For the final, we use work-energy:

$$\begin{aligned} W &= \Delta E_k + \cancelto{0}{\Delta E_p} + \cancelto{0}{\Delta E_s} \\ W &= -FL \\ \Delta E_k &= \frac{1}{2} m_B (\cancelto{0}{v_{B3}^2} - v_{B2}^2) \\ \Rightarrow -FL &= -\frac{1}{2} m_B v_{B2}^2 \text{where :} \\ F &= \mu_k N \\ N &= m_B g \end{aligned}$$

which gives us the length in ft.

a)	0	WE	1
b)	1	IM	2
c)	2	WE	3

Table 4: Problem States for Example 39

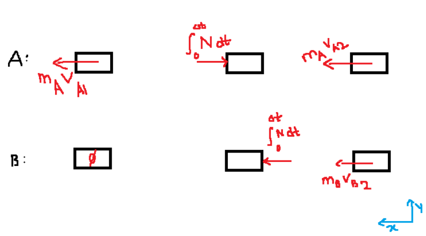

Figure 50: Example 39 IMD

3/05: Rigid Bodies Introduction

We are ignoring mass (gravitational force) since it is negligible, while we are ignoring friction because it is convenient. Neglecting friction, such as rock skipping, provides the notion that things will not slow down, even though it does. On the exam, there will be no angled impulse.

For exam 2, major concepts are:

1. Newton's Second Law, in N-T and C-P coordinates. They will be in-plane and out-of-plane (where something revolves in a circle in top view)
2. Work energy, including springs
3. Impulse momentum, including coefficient of restitution

Use Newton's Second Law (N2L) when we are looking at a single instant. We use Impulse Momentum when the time span is very small. The problems that we will have are:

1. Work Energy combined with Impulse Momentum,
2. Work Energy combined with N2L (in-plane),
3. and N2L out-of-plane.

Only one of them will be one numeric, in which the algebra is easy. In the other problems, we have to make a closed set of equations, and write what the units of the final answers will be. One of the problem is from the example problem in class, another is from the homework, and another is something that is from neither.

For 3-D problems, we will obviously be drawing 2D. We use N-Ts for N2L when we are looking at perspective of the particle, and we are not given things like $\dot{r}$ and $\dot{\theta}$.

We are entering 3rd quadrant in the class.

	Particles	Rigid Bodies
Kinematics	Exam 1	Exam 3
Kinetics	Exam 2	Exam 4

Table 5: Table of Course Topic we are in

A body that is purely translating is like a particle. Rigid bodies have translation and rotation. Rotation can be considered as rotating about a point on the body. The velocities are also on points on the body. We can combine the rotation and translation as velocity vectors going different directions on different paths of the particle (see figure 51). We will only do 2D rigid bodies.

Points do not have angular velocities. Each body has an angular velocity, the whole object has the same angular velocity. The distance vector between A and B is noted as $\vec{r}_{B/A}$. If angular velocity is zero, velocity

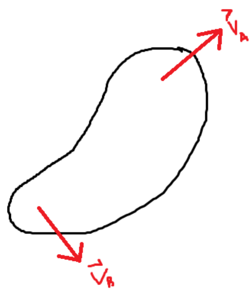

Figure 51: Rigid Body Velocity Vectors

of B and A are the same; otherwise they are different. Since we are having 2 dimensional rigid body dynamic problems, our angular velocity will always be

$$\vec{\omega} = w\hat{k} \tag{66}$$

$$\vec{\alpha} = \alpha\hat{k} \tag{67}$$

The physical quantities of $\vec{v}, \vec{r}, \vec{w}$ are related by the cross product:

$$\vec{v}_{B/A} \equiv \vec{w} \times \vec{r}_{B/A} \tag{68}$$

Using relative motion, the velocity bridge equation is

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A} \tag{69}$$

According to the equation, the velocity at two points can only differ if there is $\vec{\omega}$ and $\vec{r}$. For the acceleration vector, it is

$$\vec{v}_B = \vec{a} + \vec{\alpha} \times \vec{r}_{B/A} + \vec{\omega} \times \frac{\mathrm{d}\vec{r}_{B/A}}{\mathrm{d}t} \tag{70}$$

$$= \vec{a} + \vec{\alpha} \times \vec{r}_{B/A} - \omega^2 \vec{r}_{B/A} \tag{71}$$

The terms in the above equation are translation, tangential acceleration, and normal acceleration. Together, it is the acceleration bridge equation. Exam 3 will be equations 71 and 69.

The steps to solve rigid body problems are:

1. Bridge all the way from one support to the other. In professor's words, bridge across whole structure, and stop at shared points.
2. Define all vectors that appear in the bridge.
3. Substitute and cross product party.

We will be using Cartesian, since it is relative motion. If we do not know the direction of vector before doing the math, draw the vector in the positive direction. If math comes negative, that means it is in opposite direction.

Example 44

If roller A moves to the right with a constant velocity of 3m/s, determine the angular velocity of the link and the velocity of roller B at the instant where the angle from horizontal is 30 degrees.

We are given

$$v_A = 3\,\mathrm{m/s}$$
$$\theta = 30^\circ$$
$$L = 1.5\,\mathrm{m}$$

and asked to find $\omega, \vec{v}_B$. Our method is called bridge building. The direction (A to B, or B to A) does not matter.

$$\vec{v}_B = \vec{v}_A + \vec{\omega} \times \vec{r}_{B/A}$$

We need to write down expressions for all of those four terms above:

$$\vec{v}_A = v_A\hat{\imath}$$
$$\vec{v}_B = v_B\hat{\jmath}$$
$$\vec{\omega} = \omega\hat{k}$$
$$\vec{r}_{B/A} = (-L\cos\theta)\hat{\imath} + (L\sin\theta)\hat{\jmath}$$

For $\vec{r}_{B/A}$, we are going A to B, so we are going right to left.

$$v_B\hat{\jmath} = v_A\hat{\imath} + \omega\hat{k} \times \left[(-L\cos\theta)\hat{\imath} + (L\sin\theta)\hat{\jmath}\right]$$
$$= v_A\hat{\imath} + w(-L\cos\theta)\hat{\jmath} + w(L\sin\theta)\cdot -\hat{\imath}$$

We can split the equations into their component/scalar form:

$$\hat{\imath}: \quad 0 = v_A - wL\sin\theta$$
$$\hat{\jmath}: \quad v_A = -wL\cos\theta$$

We have two unknowns w, v_B, and two equations.

For cross products, we don't have to do the determinant of a 3D vector in this class every time. We will treat it as multiplication.

$$\hat{\imath}\,\hat{\jmath}\,\hat{k}\,\hat{\imath}\,\hat{\jmath}\,\hat{k}$$

Multiplying two above terms left to right is positive. Opposite is negative. As an example, $\hat{\imath} \times \hat{\jmath} = \hat{k}$, and $\hat{\jmath} \times \hat{\imath} = -\hat{k}$.

When we talk about acceleration, we will have

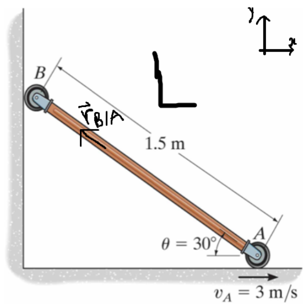

Figure 52: Example 44

4 equations and 4 unknowns. We will name points using numbers, rather than letters.

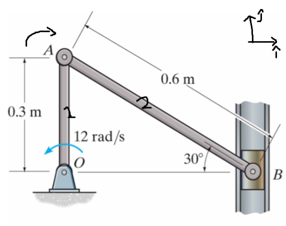

Figure 53: Example 45

Example 45

If crank OA rotates with an angular velocity of 12 rad/s, determine the velocity of piston B.

We are given

$$\begin{aligned} \omega_1 &= 12\,\text{rad/s} \\ L_1 &= 0.3\,\text{m} \\ L_2 &= 0.6\,\text{m} \\ \theta &= 30^\circ \end{aligned}$$

and are asked to find v_B. We are assuming that the body goes clockwise. The point O cannot move, so its velocity is zero.

$$\begin{aligned} \vec{v}_A &= \overset{0}{\cancel{\vec{v}_0}} + \vec{\omega}_1 \times \vec{r}_{A/O} \\ \vec{v}_B &= \vec{v}_A + \vec{\omega}_2 \times \vec{r}_{B/A} \\ &= \vec{\omega}_1 \times \vec{r}_{A/O} + \vec{\omega}_2 \times \vec{r}_{B/A} \end{aligned}$$

We will do all the bridges first, and then the cross products. Our terms are:

$$\begin{aligned} \vec{v}_B &= v_B \hat{j} \\ \vec{\omega}_1 &= \omega_1 \hat{k} \\ \vec{\omega}_2 &= \omega_2 \hat{k} \\ \vec{r}_{A/O} &= L_1 \hat{j} \\ \vec{r}_{B/A} &= (L_2 \cos\theta)\hat{i} + (-L_2 \sin\theta)\hat{j} \end{aligned}$$

Now the cross products:

$$v_B \hat{j} = \omega_1 \hat{k} \times L_1 \hat{j} + \omega_2 \hat{k} \times \left[(L_2 \cos\theta)\hat{i} + (-L_2 \sin\theta)\hat{j} \right]$$
$$v_B \hat{j} = \omega_1 L_1 (-\hat{i}) + \omega_2 (L_2 \cos\theta)\hat{j} + \omega 2(-L_2 \sin\theta)(-\hat{i})$$

Splitting the equations to scalar, we have:

$$\begin{aligned} \hat{i}: &\quad 0 = -\omega_1 L_1 + \omega_2 L_2 \sin\theta \\ \hat{j}: &\quad v_B = \omega_2 L_2 \cos\theta \end{aligned}$$

and solving, we get $\omega_2 = 12\,\text{rad/s}$ and $v_B = 6.24\,\text{m/s}$.

This will be not on the exam.

4/12: Rigid Body Examples

The exams have not been graded yet. There are only two main equations for rigid bodies: the velocity and acceleration bridge.

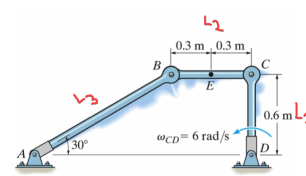

Figure 54: Example 46

Example 46

If link CD has an angular velocity of 6 rad/s determine the angular velocities of links AB and BC. Determine the velocity of point E.

We are given:

$$\begin{aligned}
\omega_1 &= 6\,\text{rad/s}\\
\theta &= 30^\circ\\
L_1 &= 0.6\,\text{m}\\
L_2 &= 0.6\,\text{m}\\
L_3 &= \frac{L_1}{\sin\theta}
\end{aligned}$$

We want to describe an internal point E. We are going to go from one structure to another. This should take us 3 steps. The order that we do this doesn't matter. Our equation for points CD is:

$$\begin{aligned}
\vec{v}_C &= \cancelto{0}{\vec{v}_D} + \vec{\omega}_1 \times \vec{r}_{C/D}\\
\vec{v}_B &= \vec{v}_C + \vec{\omega}_2 \times \vec{r}_{B/C}\\
&= \vec{\omega}_1 \times \vec{r}_{C/D} + \vec{\omega}_2 \times \vec{r}_{B/C}\\
\cancelto{0}{\vec{v}_A} &= \vec{v}_B + \vec{\omega}_3 \times \vec{r}_{A/B}\\
&= \vec{\omega}_1 \times \vec{r}_{C/D} + \vec{\omega}_2 \times \vec{r}_{B/C} + \vec{\omega}_3 \times \vec{r}_{A/B}
\end{aligned}$$

Velocity at D is zero because it is a stationary point. Points have linear velocities. All of our $\vec{\omega}$ vectors points in the $\hat{k}$ direction. The directional vectors depend on where they are going:

$$\begin{aligned}
\vec{r}_{C/D} &= L_1\hat{j}\\
\vec{r}_{B/C} &= -L_2\hat{j}\\
\vec{r}_{A/B} &= (-L_3\cos\theta)\hat{i} + (-L_3\sin\theta)\hat{j}
\end{aligned}$$

Combining all these equations, we get:

$$\begin{aligned}
\vec{0} &= \omega_1\hat{k} \times L_1\hat{j} + \omega_2\hat{k} \times (-L_2)\hat{i}\ldots\\
&+ \omega_3\hat{k} \times [(-L_3\cos\theta)\hat{i} + (-L_3\sin\theta)\hat{j}]\\
&= -\omega_1 L_1\hat{i} - \omega_2 L_2\hat{j} - \omega_3 L_3\cos\theta\hat{j} + \omega_3 L_3\sin\theta\hat{i}
\end{aligned}$$

Which can be broken down to the equations:

$$\begin{aligned}
\hat{i}:&\qquad 0 = -\omega_1 L_1 + \omega_3 L_3\sin\theta\\
\hat{j}:&\qquad 0 = -\omega_2 L_2 - \omega_3 L_3\cos\theta
\end{aligned}$$

which have the units of radians per second. To find the velocity at E, we should bridge to E:

$$\begin{aligned}
\vec{v}_E &= \vec{v}_C + \vec{\omega}_2 \times \vec{r}_{E/C}\\
&= \vec{\omega}_1 \times \vec{r}_{C/D} + \vec{\omega}_2 \times \vec{r}_{E/C}\\
&= \omega_1\hat{k} \times L_1\hat{j} - \omega_2\hat{k} \times \frac{L_2}{2}\hat{i}\\
&= \boxed{-\omega_1 L_1\hat{i} - \omega_2\frac{L_2}{2}\hat{j}}
\end{aligned}$$

The next exam will be repetitive use of the above equations.

Rolling is weird. Friction is needed to roll. Generally, actual friction is not at its maximum for rolling. Friction is what allows a car to move towards the direction of travel.

$$F \leq \mu_s N \,(\text{uncertain}) \tag{72}$$

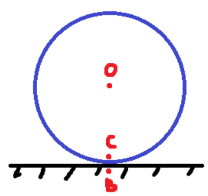

Figure 55: Rolling Ball

In a bowling alley, the lanes are very oiled, hence the balls slide. But when we hear rumbling, it's rolling. Rolling implies no sliding.

In figure 55, C is the contact point on the wheel as it touches the ground. B is the contact part on ground touching the wheel. At any given time, they are aligned. Velocity of C matches velocity B. For the fraction of a second, that velocity is zero (if the ground isn't moving). The acceleration, however, is not zero.

If it is rolling and not sliding, velocity of C matches velocity B.

In tires, there's no single contact. There's a patch of area that makes contact with the road. The ground also deforms down, so a rolling object is constantly rolling uphill in a way. The force from ground to the rolling object is slightly greater in front of the object than behind.

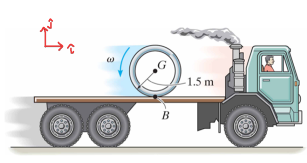

Figure 56: Example 47

Example 47

At the instant shown, the truck travels to the right at 3m/s while the pipe rolls counterclockwise at 8 rad/s without slipping at B. Determine the velocity of the pipe's center G.

Given

$$v_B = 3\,\text{m/s}$$
$$R = 1.5\,\text{m}$$
$$\omega = 8\,\text{rad/s}$$

and asked to find $\vec{v}_G$ (the center of the rolling object). Since this is rolling:

$$\begin{aligned}
\vec{v}_C &= \vec{v}_B \\
&= v_B\hat{i} \\
\vec{v}_G &= \vec{v}_C + \vec{\omega} \times \vec{r}_{G/C} \\
\vec{v}_G &= v_B\hat{i} + \omega\hat{k} \times R\hat{j} \\
&= (v_B - \omega R)\hat{i}
\end{aligned}$$

If the object was rolling on the ground, $v_B = 0$.

Radians are dimensionless. Multiplying this with something else that has dimensions will result in that dimension.

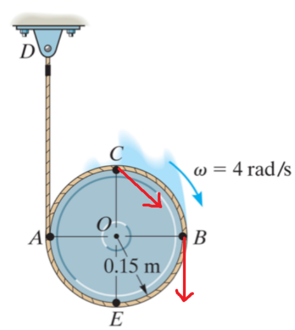

Figure 57: Example 48

Example 48

At the instant shown, the disk is rotating at an angular velocity of 4 rad/s. Determine the velocities of points A, B, and C

We are asked to find $\vec{v}_A, \vec{v}_B, \vec{v}_C$. A yo-yo would climb up when it reaches the ground. We are assuming that the rope cannot stretch. The velocity of the contact point that touches the vertical segment of the rope is zero. Therefore $\vec{v}_A = \vec{0}$. We are given

$$R = 0.15\,\text{m}$$
$$\omega = -4\,\text{rad/s}$$
$$\vec{\omega} = -\omega\hat{k}$$

Solving for point, B, we get::

$$\begin{aligned}
\vec{v}_B &= \cancelto{0}{\vec{v}_A} + \vec{\omega} \times \vec{r}_{B/A} \\
&= -\omega\hat{k} \times 2R\hat{i} \\
&= -2\omega R\hat{j} \\
&= \boxed{-1.2\,\text{m/s}\hat{j}}
\end{aligned}$$

For C, we can bridge with A:

$$\begin{aligned}
\vec{v}_C &= \cancelto{0}{\vec{v}_A} + \vec{\omega} \times \vec{r}_{C/A} \\
&= -\omega\hat{k} \times [R\hat{i} + R\hat{j}] \\
&= -\omega R\hat{j} + \omega R\hat{j} \\
&= \boxed{0.6\hat{i} - 0.6\hat{j}\ [\text{m/s}]}
\end{aligned}$$

4/14: More Problems and Rolling

There are three boundaries. Two of them are pins, and tracks. The tracks can be curved or uncurved.

When doing these problems, we have to make sure to either use clock wise, or counterclockwise with negative value (this is assuming that something is rotating clockwise already.)

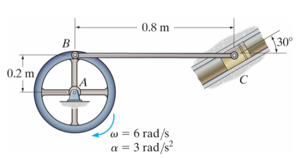

Figure 58: Example 49

Example 49

At the instant shown, wheel A rotates with an angular velocity of 6 rad/s and an angular acceleration of 3 rad/s 2 as shown. Determine the acceleration of piston C.

We are given

$$\omega = -6\,\text{rad/s}$$
$$\alpha = -3\,\text{rad/s}^2$$
$$R = 0.2\,\text{m}$$
$$L = 0.8\,\text{m}$$
$$\theta = 30^\circ$$

A double cross product is just a square. We have to make vector bridge equations. Because these are vectors, we have two scalar equations in the $\hat{i}$ and $\hat{j}$ direction. If the track was curved, we would also have to deal with ρ. The bridge equations give us two equations, but we can have more unknowns than that. The track has a motion in N-Ts, and we can convert them to cartesian. The scalar answer would be sufficient because we defined the vectors already.

General steps taken:

1. For linkages that are named with letters, rename them with numbers.
2. Convert direction vectors to be in the same direction (e.g. make all of them counterclockwise).
3. Write the knowns and any conversions
4. Write the bridge equations (and combine them)
5. Define each of the variables in the bridge equations
6. Cross product party
7. Obtain the scalar equations (generally 4)
8. Solve

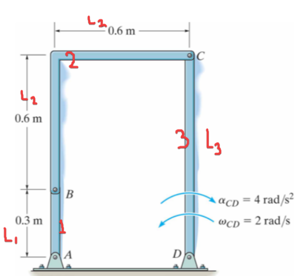

Figure 59: Example 50

Example 50

Determine the angular acceleration of link AB if link CD has the angular velocity and angular acceleration shown

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We are given:

$$\begin{aligned} L_1 &= 0.3\,\mathrm{m} \\ L_2 &= 0.6\,\mathrm{m} \\ L_3 &= 0.9\,\mathrm{m} \\ \omega_3 &= 2\,\mathrm{rad/s} \\ \alpha_3 &= -4\,\mathrm{rad/s^2} \end{aligned}$$

The bridge equations:

$$\begin{aligned} \vec{a}_B &= \cancelto{0}{\vec{a}_A} + \vec{\alpha}_1 \times \vec{r}_{B/A} - \omega_1^2 \vec{r}_{B/A} \\ \vec{a}_C &= \vec{a}_B + \vec{\alpha}_2 \times \vec{r}_{C/B} - \omega_2^2 \vec{r}_{C/B} \\ &= \vec{\alpha}_1 \times \vec{r}_{B/A} - \omega_1^2 \vec{r}_{B/A} + \vec{\alpha}_2 \times \vec{r}_{C/B} - \omega_2^2 \vec{r}_{C/B} \\ \cancelto{0}{\vec{a}_D} &= \vec{a}_C + \vec{\alpha}_3 \times \vec{r}_{D/C} - \omega_3^2 \vec{r}_{D/C} \\ &= \vec{\alpha}_1 \times \vec{r}_{B/A} - \omega_1^2 \vec{r}_{B/A} \\ &\cdots + \vec{\alpha}_2 \times \vec{r}_{C/B} - \omega_2^2 \vec{r}_{C/B} \\ &\cdots + \vec{\alpha}_3 \times \vec{r}_{D/C} - \omega_3^2 \vec{r}_{D/C} \end{aligned}$$

Note that the pins at the base do not have linear motions. There is a template between the acceleration bridges and the velocity bridge, where

$$\begin{aligned} \vec{a} &\to \vec{v} \\ \vec{\alpha} &\to \vec{\omega} \\ \omega^2 &\to 0 \end{aligned}$$

Therefore, we get the bridge for the velocity to be:

$$0 = \vec{\omega}_1 \times \vec{r}_{B/A} + \vec{\omega}_2 \times \vec{r}_{C/B} + \vec{\omega}_3 \times \vec{r}_{D/C}$$

All of our $\vec{\omega}_i, \vec{\alpha}_i$ have the form $\omega_i \hat{k}, \alpha_i \hat{k}$ respectively. The displacement vectors are:

$$\begin{aligned} \vec{r}_{B/A} &= L_1 \hat{j} \\ \vec{r}_{C/B} &= L_2 \hat{i} + L_2 \hat{j} \\ \vec{r}_{D/C} &= -L_3 \hat{j} \end{aligned}$$

We combine the above definitions into the bridge equations. We will get four scalar equations. The four equations are obtained by beraking down the vector equations into scalars.

$$\begin{aligned} &a\hat{i} && 0 = -\alpha_1 L_1 - \alpha_2 L2 - \omega_2^2 L_2 + \alpha_3 L_3 \\ &a\hat{j} && 0 = -\omega_1^2 L_1 - \alpha_2 L2 - \omega_2^2 L_2 + \omega_3^2 L_3 \\ &v\hat{i} && 0 = -\omega_1 L_1 - \omega_2 L_2 + \omega_3 L_3 \\ &v\hat{j} && 0 = \omega_2 L_2 \end{aligned}$$

The $\omega_2 = 0$ because the top left part does not rotate. The answers are

$$\begin{aligned} \omega_2 &= 0\,\mathrm{rad/s} \\ \omega_1 &= 6\,\mathrm{rad/s} \\ \alpha_1 &= ? \\ \alpha_2 &= ? \end{aligned}$$

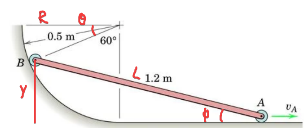

Figure 60: Example 51

Example 51

If the velocity of point A is 3m/s to the right and is constant for an interval including the position shown, determine the acceleration of point B and the angular acceleration of the bar.

$$\begin{aligned} R = R\sin\theta + y \Rightarrow y &= R - R\sin\theta \\ \phi = \arcsin\left(\frac{y}{L}\right) \end{aligned}$$

We will do a bit of rolling theory now. This is rolling is weird 2.0. When we have a velocity problem, the velocity at the contact points are the same. If something is moving on still ground (e.g. tire wheels), the velocities would be zero. The velocity of the center is

$$\vec{v}_G = \cancelto{0}{\vec{v}_C} + \omega \hat{k} \times R\hat{j} \tag{73}$$

$$= -\omega R \hat{i} \tag{74}$$

The contact point goes up, down, touches the ground, then up, down, touches the ground, and so on. The acceleration of the center and the contact

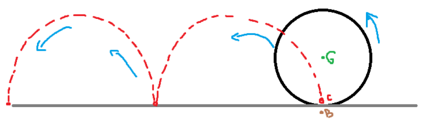

Figure 61: Contact Points Rolling Over

point are:

$$\vec{a}_G = -\alpha R\hat{i} \tag{75}$$

$$\vec{a}_C = \cancelto{0}{-\alpha R\hat{i}} + \cancelto{0}{\alpha\hat{k} \times (-R)\hat{j}} - \omega^2(-R)\hat{j} \tag{76}$$

$$= \omega^2 R\hat{j} \tag{77}$$

The key realization is that the acceleration of the contact point is going to match the acceleration of the ground plus the normal acceleration to the center. Our two big equations for the rolling boundary condition is

$$\vec{a}_C = \vec{a}_B + \omega^2 r\hat{u}_N \tag{78}$$

$$\vec{v}_C = \vec{v}_B \tag{79}$$

where the $\hat{u}_N$ is perpendicular to whatever the rolling object is touching.

If it is pinned, velocity and acceleration is zero. If it is in a track, velocity is parallel to the track and acceleration is perpendicular to the surface(if any). If it is rolling, then velocity of the contact is velocity of the ground, while the acceleration is the acceleration of the ground plus the normal acceleration.

Exam 2 was handed out. Exam 3 is next week on Friday.

4/19: More rolling

We won't have problems that have too many geometric problems. We will have rolling problems. There are two tofu rules: one with and without the ground in motion.

We have three boundary conditions for rigid bodies:

1. Pin connected, where $\vec{v}_A = \vec{0}$ and $\vec{a}_A = \vec{0}$.
2. The moving part is in a track.
3. Rolling, where object's rolling point in contact with ground has same velocity as the ground.

Our general method is to bridge from known point to the unknown point. Rule of thumb: If we have an acceleration problem, we have to deal with velocity as well. A rack is gear that has an infinite radius. The pinion is the one with the smallest radius.

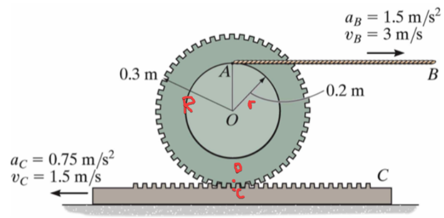

Figure 62: Example 52

Example 52

At the instant shown, cable and rack have a velocities and accelerations as shown. Determine the angular acceleration of the gear at this instant.

We know v_B, v_C, a_B, a_C, r, R. We are asked to find the α. First, as usual, write the bridge equations:

$$\vec{v}_A = \vec{v}_D + \vec{\omega} \times \vec{r}_{A/D}$$

$$\vec{a}_A = \vec{a}_D + \vec{\alpha} \times \vec{r}_{A/D} - \omega^2\vec{r}_{A/D}$$

and then define the variables

$$\vec{v}_A = v_B\hat{i}$$

$$\vec{v}_D = -v_C\hat{i}$$

$$\vec{a}_A = a_B\hat{i} + (-\omega^2 r)\hat{j}$$

$$\vec{a}_D = -a_C\hat{i} + (\omega^2 R)\hat{j}$$

$$\vec{\omega} = \omega\hat{k}$$

$$\vec{\alpha} = \alpha\hat{k}$$

$$\vec{r}_{A/D} = (R + r)\hat{j}$$

An object spinning with a rope behaves as if it is rolling on the rope. Substitute these into our equations:

$$v_B\hat{i} = (-v_C)\hat{i} + \omega\hat{k} \times (R + r)\hat{j}$$

$$a_B\hat{i} + (-\omega^2 r)\hat{j} = (-a_C)\hat{i} + (\omega^2 R)\hat{j} + \alpha\hat{k} \times (R + r)\hat{j} - \omega^2(R + r)\hat{j}$$

and breaking these into their components for v and a:

$$\hat{i}: \quad v_B = -v_C - \omega(R + r)$$

$$\hat{j}: \quad 0 = 0$$

$$\hat{i}: \quad a_B = -a_C - \alpha(R + r)$$

$$\hat{j}: \quad -\omega^2 r = \omega^2 R - \omega^2(R + r)$$

Note that the last equation also cancels to zero. We are left with two unknowns: α, ω.

We should not rotate our coordinate system. There will 100% be rolling problems on the exam. Example 53 tests a lot of things.

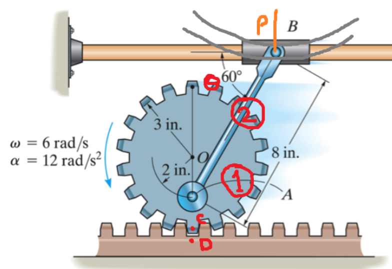

Figure 63: Example 53

Example 53

At a given instant, the gear has an angular motion as shown. Determine the acceleration of point B

We have two rigid bodies: the rolling gear, and the bar that's connecting the gear with the horizontal bar. We are given:

$$\omega_1 = 6\,\mathrm{rad/s}$$
$$\alpha_1 = 12\,\mathrm{rad/s^2}$$

The other end of the structure is our defined point C, which mates with the horizontal gear. We are asked to find a_B. A gear problem is how we force the notion that the rolling object is not slipping. Bridge:

$$\vec{a}_A = \vec{a}_C + \vec{\alpha}_1 \times \vec{r}_{A/C} - \omega_1^2 \vec{r}_{A/C}$$
$$\vec{a}_B = \vec{a}_A + \vec{\alpha}_2 \times \vec{r}_{B/A} - \omega_2^2 \vec{r}_{B/A}$$
$$\Rightarrow \vec{a}_B = \vec{a}_C + \vec{\alpha}_1 \times \vec{r}_{A/C} - \omega_1^2 \vec{r}_{A/C} \ldots$$
$$\cdots + \vec{\alpha}_2 \times \vec{r}_{B/A} - \omega_2^2 \vec{r}_{B/A}$$
$$\vec{v}_B = \vec{v}_C + \vec{\omega}_1 \times \vec{r}_{A/C} + 0 \ldots$$
$$\cdots + \vec{\omega}_2 \times \vec{r}_{B/A} + 0$$

Note that it is easy to get $\vec{v}$ from $\vec{a}$ since we are only swapping accelerations for velocities.

Our variables are:

$$\vec{\alpha}_1 = \alpha_1 \hat{k}$$
$$\vec{\alpha}_2 = \alpha_2 \hat{k}$$
$$\vec{\omega}_1 = \omega_1 \hat{k}$$
$$\vec{\omega}_2 = \omega_2 \hat{k}$$
$$\vec{r}_{A/C} = (R - r)\hat{j}$$
$$\vec{r}_{B/A} = (L\cos\theta)\hat{i} + (L\sin\theta)\hat{j}$$
$$\vec{a}_B = a_B \hat{i} + \left(\frac{v_B^2}{\rho}\right)\hat{j}$$
$$\vec{v}_B = v_B \hat{i}$$
$$\vec{a}_C = \vec{0} + (\omega_1^2 R)\hat{j}$$
$$\vec{v}_C = \vec{0}$$

Note that $\rho \to \infty$. To solve this, we first have to do the cross product, which would give us velocities and accelerations in the $\hat{i}$ and $\hat{j}$ directions (4 equations). The unknowns are $v_B, a_B, \alpha_2, \omega_2$.

For the exam, there will be 1 pure velocity problem, 1 acceleration problem (hence we also have to do the velocity), and 1 problem that has rolling problem. One of them will be numerical (it will be the velocity problem).

We will now transition to rigid body kinetics. These are NOT on the upcoming exam. We had:

1. Kinematic Points (Exam 1)
2. Kinetic Points (Exam 2)
3. Kinematic Rigid Bodies (Exam 3, the upcoming exam)
4. Kinetics Rigid Bodies (New topic, exam 4)

We have 3 types of kinetics: Newton's Second Law, work-energy, impulse-momentum. We will not be doing impulse-momentum for rigid bodies, because impacts don't keep objects 'rigid'.

For rigid bodies, it has forces, and moments. We consider the force based off of the center of mass. We track the motion for the center of mass. For rotations, we also do it about the center of mass. The two equations for Newton's second law we obtain are:

$$\sum F = m\vec{a}_G \tag{80}$$
$$\sum M_G = I_G \vec{\alpha} \tag{81}$$
$$\tag{82}$$

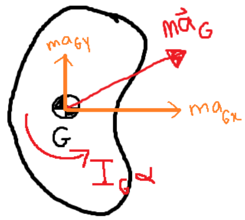

Figure 64: Force on rigid body

Example 54

The slender rod has a mass of 10kg and the sphere has a mass of 15kg. Locate the center of gravity. Determine the moment of inertia about the center of gravity. Determine the radius of gyration.

We find the center of gravity for each object, and then average it.

$$\frac{m_1 y_1 + m_2 y_2}{m_1 + m_2} = \bar{y}$$

To obtain the I_G, we use parallel axis theorem:

$$I_G = I_1 + m_1 d_1^2 + I_2 + m_2 d_2^2$$

Note that the I_n is the inertia about the object's center of mass. The d is the distance from their center of mass to the point of concern (in this case, the center of gravity for the entire object).

The radius of gyration is the radius of a hoop, if the hoop had equivalent mass and inertia as our complex object. The radius of gyration is given as

$$K_G = \sqrt{\frac{I_G}{m}} \tag{83}$$

We have to do the problem in Cartesian.

Reminder: **EXAM ON FRIDAY**. We have homework due on the 20th.

4/26: Rigid Bodies

We will do a lot of example problems this week.

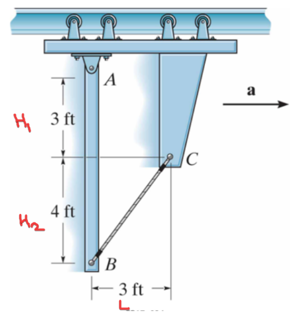

Figure 65: Example 55

Example 55

The 20lb link AB is pinned to a moving frame at A and held in a vertical position by means of a string BC which can support a maximum tension of 10lbs. Determine the maximum acceleration of the frame without breaking the string. What are the components of the reaction at pin A.

We are looking for a_{max} and reaction forces at A. Note that the object can be moved to the right only using the string, because the string works in tension, not compression. We are given:

$$H_1 = 4\,\text{ft}$$
$$H_2 = 3\,\text{ft}$$
$$L_1 = 3\,\text{ft}$$
$$\theta = \arctan\left(\frac{H_1}{L_1}\right)$$
$$m = \frac{20}{32.2}\,\text{slug}$$
$$T_max = 10\,\text{lbs}$$

We have to draw a free body diagram (FBD) and a mass acceleration diagram (MAD). We should not make assumptions in our figures, but in our equations.

$$A_x + T\cos\theta = ma_{Gx}$$
$$A_y - mg + T\sin\theta = ma_{Gy}$$
$$T\cos\theta\left(\frac{H_1 + H_2}{2}\right) A_x \left(\frac{H_1 + H_2}{2}\right) = I_G\alpha$$

We should never treat moment of inertia as the unknown in our equation. We should break it down to mass and distance instead.

$$I_G = \frac{m(H_1 + H_2)^2}{12}$$

We have 3 equations and 7 unknowns. We can assume our tension is the maximum. Since the bar always remains vertical, $\alpha = 0$. Since motion is only in the horizontal direction, $a_{Gx} = a, a_{Gy} = 0$.

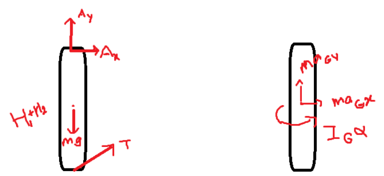

Figure 66: FBD and MAD for Example 55

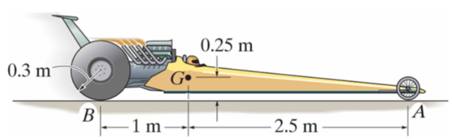

Figure 67: Example 56

Example 56

The rear-wheel drive dragster has a mass of 1500 kg and a center of mass at G. If no slipping occurs, determine the friction force which must be developed at the back tires to cause an acceleration of 6m/s2 . What are the normal forces of each wheel on the ground?. Neglect the mass of the tires and assume the front wheels are "free to roll".

Two huge assumptions are that the mass of the tires are negligible, and that wheels are freely rolling. We give a torque to the tires, but we won't be dealing with it today. We are given:

$$m = 1500\,\mathrm{kg}$$
$$L_B = 1\,\mathrm{m}$$
$$L_A = 2.5\,\mathrm{m}$$
$$R = 0.3\,\mathrm{m}$$
$$H = 0.25\,\mathrm{m}$$
$$a = 6\,\mathrm{m/s^2}$$

We have double the friction and normal forces because there are two tires.

$$2F = ma_{Gx}$$
$$2N_B + 2N_A - mg = ma_{Gy}$$
$$2N_AL_A - 2N_BL_B + 2FH = I_G\alpha$$

Extra information generally comes from friction and springs. Since the vehicle is moving horizontally, acceleration components are $a_{Gx} = a, a_{Gy} = 0$, and since it is not rotating (or doing a wheelie) $\alpha = 0$. We have 6 equations and 6 unknowns. Solving this yields:

$$F = 4500\,\mathrm{N}$$
$$N_A = 1780\,\mathrm{N}$$
$$N_B = 5576.8\,\mathrm{N}$$
$$a_{Gx} = 6\,\mathrm{m/s^2}$$
$$a_{Gy} = 0\,\mathrm{m/s^2}$$
$$\alpha = 0\,\mathrm{rad/s^2}$$

We do not need I_G because $I_G\alpha = 0$.

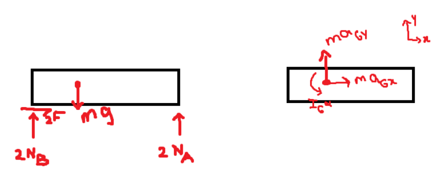

Figure 68: Example 56 FBD MAD

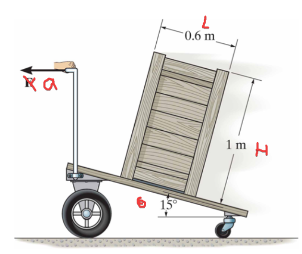

Figure 69: Example 57

Example 57

The uniform crate has a mass of 50kg and rests on the cart having an inclined surface. Determine the smallest acceleration that will cause the crate to either slip or tip relative to the cart. The coefficient of static friction between the crate and the cart is 0.5.

We are given

$$L = 0.6\,\text{m}$$
$$H = 1\,\text{m}$$
$$m = 50\,\text{kg}$$
$$\theta = 15^\circ$$

If it is about to slide, then static friction is at the maximum. The crate tips at a certain point, when the center of gravity is just over the edge. In our x component:

$$-F\cos\theta + N\sin\theta = ma_{Gx}$$

and in the y component:

$$F\sin\theta + N\cos\theta - mg = ma_{Gy}$$

and finally in the z component:

$$-F\frac{H}{2} + Ne = I_G\alpha$$

The e is the small indeterminate distance that the CG moves by just before moving. Since the object is 'almost' slipping and tipping, our $a_{Gx} = -a, a_{Gy} = 0, \alpha = 0$. The crate can not both slip and tip. If the friction force is less, then the crate will tip: $F = \mu_s N$. Solving, we would get:

$$a_1 = 2.007\,\text{m/s}^2$$

But note that the above answer is with the assumption that it is about to slip. If the crate was about to tip instead,

$$e = \frac{L}{2}$$
$$a_2 = 2.8\,\text{m/s}^2$$

Since $a_1 < a_2$, $a = a_1$ because that happens first.

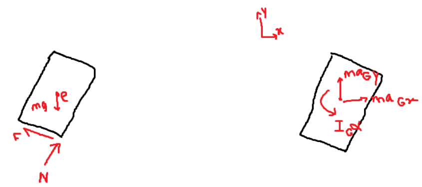

Figure 70: Example 57 FBD MAD

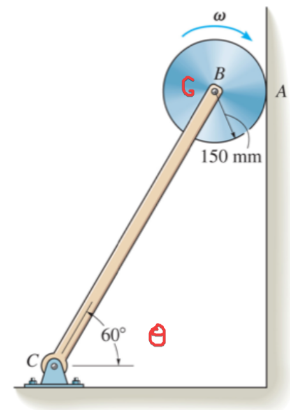

Figure 71: Example 58

Example 58

The disk has a mass of 20kg and is originally spinning at the end of the strut with an angular velocity of 60 rad/s as shown. If it is then placed against the wall (for which the coefficient of friction is 0.3), determine the time required for the motion to stop. What is the force in the strut BC during this time?

We will have to find the angular acceleration to find the time to stop. We will also find the force on the bar even though we are not asked for. We are given $m, R, \omega_0, \mu_k, \theta$. We are assuming a massless bar. This is a two force member so we can treat it as either in compression or tension. The center of gravity is at B because we are treating the bar as massless. Our equations are:

$$-N - F_{BC}\cos\theta = ma_{Bx}$$
$$-mg + F - F_{BC}\sin\theta = ma_{By}$$
$$FR = I_B\alpha$$

We have 7 unknowns. Our friction is $F = \mu_k N$. Since we don't have any linear accelerations, $a_{Bx} = a_{By} = 0$. Since the angular velocity is a derivative of acceleration:

$$\omega = \alpha t_f + C_1$$
$$= \alpha t_f - \omega_0$$

With these, we have all the equations we need to solve the problem.

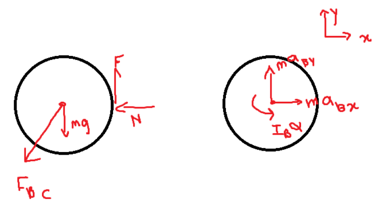

Figure 72: Example 58 FBD MAD

4/28: More problems

The homework will be due later than Sunday, since we did not cover enough example problems. We will go through as many example problems as possible.

Example 59

The cord is wrapped around the inner core of the spool. If a 5lb block B is suspended from the cord and released from rest, determine the spool's angular velocity after three seconds. The spool has a weight of 180lb and a radius of gyration around the axle of 1.25ft

---

When we are given a radius of gyration, they are giving us moment of inertia. Given:

$$m_A = \frac{180}{32.2}\,\text{slugs}$$
$$m_B = \frac{5}{32.2}\,\text{slugs}$$
$$R = 2.75\,\text{ft}$$
$$r = 1.5\,\text{ft}$$
$$k_A = 1.25\,\text{ft}$$

The radius of gyration can be used to find the inertia, since

$$I_A = m_A k_A^2$$

Impact is the only real reason to use impulse momentum. Without impact, we shouldn't use it, especially in rigid body problems. Use Newton's Second Law. We are asked to find ω_f. We need two diagrams for both of the objects. Using the diagrams, we get

$$\begin{aligned}
x:&\quad A_x = m_A a_A\\
y:&\quad A_y - m_A g - T = m_A a_{Ay}\\
z:&\quad -Tr = I_A\alpha\\
x:&\quad 0 = m_B a_{Bx}\\
y:&\quad T - m_B g = m_B a_{By}
\end{aligned}$$

We know, however, that $a_{Ax} = a_{Ay} = 0$. The acceleration at r is same of the acceleration of B. Suppose there exists a contact point C where the cord meets the inner radius. We can bridge:

$$\begin{aligned}
\vec{a}_C &= \cancelto{0}{\vec{a}_A} + \vec{\alpha}\times\vec{r}_{C/A} - \omega^2\vec{r}_{C/A}\\
\vec{a}_C &= \alpha\hat{k}\times r\hat{i} - \omega^2 r\hat{i}\\
\vec{a}_C &= \alpha r\hat{j} - \omega^2 r\hat{i}\\
a_{By} &= a_{Cy} = \alpha r\\
\omega_f &= \alpha t_f
\end{aligned}$$

Since it starts from rest, there is no initial ω.

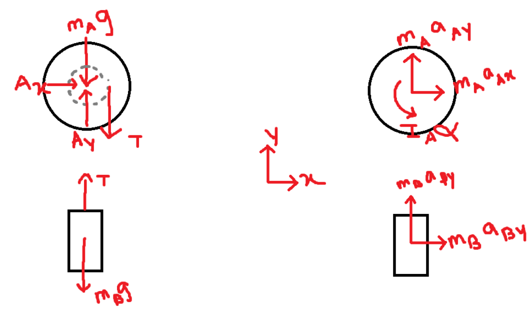

Figure 73: Example 59 FBD MAD

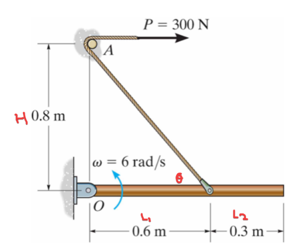

Figure 74: Example 60

Example 60

The uniform 30kg slender rod is being pulled by the cord that passes over the small smooth peg at A. If the rod has an angular velocity of 6 rad/s at this instant, determine the reactions at the pin O and the angular acceleration of the rod.

If this bar were to move it would be a work-energy problem. At this very instant though, it is a Newton's Second Law. We are given:

$$L_1 = 0.6\,\text{m}$$
$$L_2 = 0.3\,\text{m}$$
$$H = 0.8\,\text{m}$$
$$\omega = 6\,\text{rad/s}$$
$$L_B = L_1 + L_2$$
$$\theta = \arctan(\frac{H}{L_1})$$

We want the forces at O and α. We will be given simple geometries, or radius of gyration, in this class. Using the diagrams, we get:

$$x: \quad O_x - T\cos\theta = ma_{Gx}$$
$$y: \quad O_y - mg + T\sin\theta = ma_{Gy}$$
$$z: \quad T\sin\theta(L_1 - \frac{L_2}{2}) - O_y\frac{L_B}{2} = I_B\alpha$$
$$T = P$$

We bridge from O to G to obtain some unknowns:

$$\begin{aligned}\vec{a}_G &= \cancelto{0}{\vec{a}_O} + \vec{\alpha}\times\vec{r}_{G/O} - \omega^2\vec{r}_{G/O}\\ &= \alpha\hat{k}\times\frac{L_B}{2}\hat{\imath} - \omega^2\frac{L_B}{2}\hat{\imath}\\ &= \left(-\omega^2\frac{L_B}{2}\right)\hat{\imath} + \left(\frac{\alpha L_B}{2}\right)\hat{\jmath}\end{aligned}$$

which gives us

$$a_{Gx} = -\omega^2\frac{L_B}{2}$$
$$a_{Gy} = \frac{\alpha L_B}{2}$$

The angular stuff will be in rad/s.

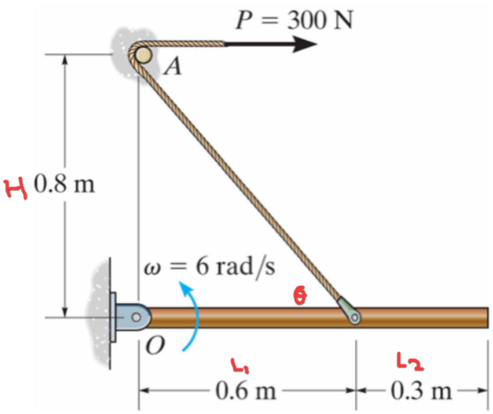

Figure 75: Example 60

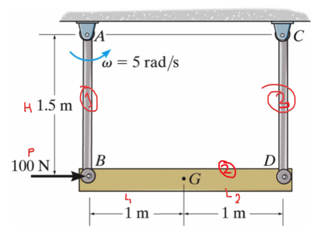

Figure 76: Example 61

Example 61

At the instant shown both rods of negligible mass swing with a counterclockwise angular velocity of 5 rad/s while the 50kg bar is subjected to the 100N horizontal force. Determine the tension developed in the rods and the angular acceleration of the rods at this instant.

Given

$$H = 1.5\,\text{m}$$
$$\omega = 5\,\text{rad/s}$$
$$P = 100\,\text{N}$$
$$L_1 = 1\,\text{m}$$
$$L_2 = 1\,\text{m}$$

and asked to find T_1, T_2, α. The bar will be perpendicular to the upper surface, therefore bar 2 will not rotate but the supports will with the same angular characteristics: $\alpha_1 = \alpha_3, \omega_1 = \omega_3, \alpha_2 = 0$. The vertical bars are of negligible mass, so we don't need to do the diagrams for those. Using the diagrams, we get:

$$
\begin{aligned}
P &= ma_{Gx} \\
T_1 + T_3 - mg &= ma_{Gy} \\
-T_1 L_1 + T_3 L_2 &= I_G \alpha_2 \\
\alpha_2 = 0\alpha_1 && = \alpha_3
\end{aligned}
$$

Since the top of the rods are stationary, we will bridge from there to the center of gravity.

$$
\begin{aligned}
\vec{a}_B &= \cancelto{0}{\vec{a}_A} + \vec{\alpha}_1 \times \vec{r}_{B/A} - \omega_1^2 \vec{r}_{B/A} \\
\vec{a}_G &= \vec{a}B + \cancelto{0}{\vec{\alpha}_2 \times \vec{r}_{G/B}} - \cancelto{0}{\omega_2^2 \vec{r}_{G/B}} \\
&= \alpha_1 \hat{k} \times (-H)\hat{j} - \omega_1^2(-H)\hat{j} \\
&= (\alpha_1 H)\hat{i} + (\omega_1^2 H)\hat{j}
\end{aligned}
$$

which yields the two additional equations:

$$
\begin{aligned}
\vec{a}_{Gx} &= \alpha_1 H \\
\vec{a}_{Gy} &= \omega_1^2 H
\end{aligned}
$$

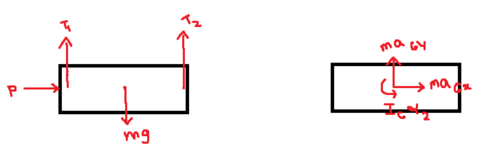

Figure 77: Example 61 FBD MAD

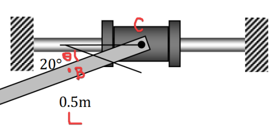

Figure 78: Example 63

Example 63

The 0.5m long 5kg uniform bar is pin connected to a 10kg collar that is free to slide on a smooth horizontal guide. The bar is initially at a 20° angle with respect to the smooth guide when it is released from rest. Determine the initial angular acceleration of the bar and the initial acceleration of the collar

---

We are given:

$$
\begin{aligned}
m_B &= 5\,\text{kg} \\
m_C &= 10\,\text{kg} \\
L &= 0.5\,\text{m} \\
\theta &= 20^\circ
\end{aligned}
$$

and asked to find $\alpha, \vec{a}_C$. Note that the bar cannot rotate. Using the diagrams, we get

$$
\begin{aligned}
-C_x &= m_C a_{Cx} \\
N - m_C g - Cy &= m_C a_{Cy} \\
C_x &= m_B a_{Bx} \\
C_y - m_B g &= m_B a_{By} \\
-C_x \frac{L}{2}\sin\theta + C_y \frac{L}{2}\cos\theta &= I_B \alpha
\end{aligned}
$$

For today, our radius of curvature is infinite for the collar, therefore $a_{Cy} = 0$. We can now bridge from the collar to B.

$$
\begin{aligned}
\vec{a}_B &= \vec{a}_C + \vec{\alpha} \times \vec{r}_{B/C} - \omega^2 \vec{r}_{B/C} \\
&= (a_{Cx})\hat{i} + \alpha\hat{k} \times \left[\left(-\frac{L}{2}\cos\theta\right)\hat{i} + \left(-\frac{L}{2}\sin\theta\right)\hat{j}\right] \\
&= \left(a_{Cx} + \alpha\frac{L}{2}\sin\theta\right)\hat{i} + \left(-\alpha\frac{L}{2}\cos\theta\right)\hat{j}
\end{aligned}
$$

Which gives us the two final unknowns:

$$
\begin{aligned}
a_{Bx} &= a_{Cx} + \frac{\alpha L}{2}\sin\theta \\
a_{By} &= -\frac{\alpha L}{2}\cos\theta
\end{aligned}
$$

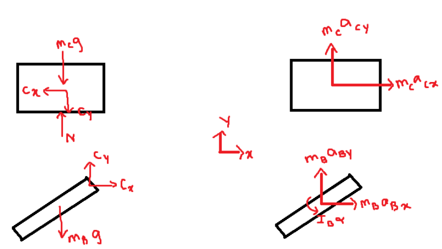

Figure 79: Example 63 FBD MAD

5/03: More rolling problems

When we do not know if an object is rolling or sliding, assume rolling. If the friction value is not possible, it implies sliding. Additionally, if every object in a diagram is at the same angle, we *can* rotate the coordinate system; only gravity will end up being non-perpendicular to any of the axis. Friction attempts to be static until it can not. For some problems, we can have multiple answers depending on what happens (e.g. friction being negative or positive.) Friction has to be less than max friction to be rolling. If an object is sliding, $F = \mu_k N$. If an object is rolling, $a_{Gx} = \alpha R$. Max friction is $F_{max} = \mu_s N$. If the pulling force of a yo-yo is extreme up, and it lifts off, N goes to zero. Often, if N is negative, it implies that the yo-yo is flying up.

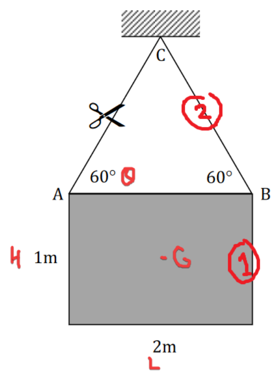

Figure 80: Example 64

Example 64

The uniform 10kg rectangular panel is suspended from point C by two thin wires A and B. If the wire A is suddenly cut, determine the tension in the wire B the instant after the wire is severed.

Do not treat this as static problem. The panel will rotate. The wire is a *massless* rigid body. If the wire was a massed bar, there would be two masses in our problem. We are given $m = 10\,\text{kg}, L = 2\,\text{m}, H = 1\,\text{m}, \theta = 60°$. We are asked to find T. Using our diagrams:

$$
\begin{aligned}
x: &\quad -T\cos\theta = ma_{Gx} \\
y: &\quad -mg + T\sin\theta = ma_{Gy} \\
z: &\quad T\cos\theta\left(\frac{H}{2}\right) + T\sin\theta\left(\frac{L}{2}\right) = I_G\alpha_1
\end{aligned}
$$

and then bridge from the ceiling, since acceleration there is zero.

$$
\begin{aligned}
\vec{a}_B &= \cancelto{0}{\vec{a}_C} + \vec{\alpha}_2 \times \vec{r}_{B/C} - \cancelto{0}{\omega_2^2\vec{r}_{B/C}} \\
\vec{a}_G &= \vec{a}_B + \vec{\alpha} \times \vec{r}_{G/B} - \cancelto{0}{\omega_1^2\vec{r}_{G/B}} \\
&= \vec{\alpha}_2 \times \vec{r}_{B/C} + \vec{\alpha}_1 \times \vec{r}_{G/B} \\
&= \alpha_2\hat{k} \times \left[(L\cos\theta)\hat{i} + (-L\sin\theta)\hat{j}\right] + \alpha_1\hat{k} \times \left[\left(-\frac{L}{2}\right)\hat{i} + \left(-\frac{H}{2}\right)\hat{j}\right] \\
&= \left(\alpha_2 L\sin\theta + \alpha_1\frac{H}{2}\right)\hat{i} + \left(\alpha_2 L\cos\theta - \alpha_1\frac{L}{2}\right)\hat{j}
\end{aligned}
$$

At the instant the cable is cut, there is no angular velocity. Additionally, the above equation brings a fifth new unknown α_2, but two equations. I_G can be obtained from the diagram.

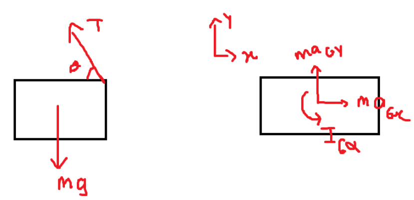

Figure 81: Example 64 FBD and MAD

Example 65

Two slender bars, each with a mass of m, are hinged at B and pinned at C. The two bars are at rest when a force F is suddenly applied at A. Determine a closed set of equations that could be used to find the initial angular accelerations of the bars.

We are given m, L, F, and asked to find α_1, α_2. We obtain 6 equations from the diagrams, and four more equations from bridging, to obtain

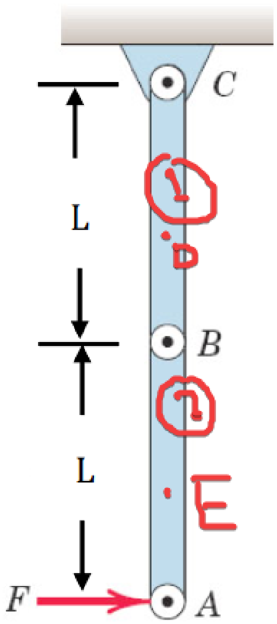

Figure 82: Example 65

10 unknowns.

$$
\begin{aligned}
x:&\quad C_x + B_x = ma_{Dx} \\
y:&\quad C_y + B_y - mg = ma_{Dy} \\
z:&\quad B_x \cdot \frac{L}{2} - C_x \cdot \frac{L}{2} = I_D \alpha_1 \\
x:&\quad -B_x + F = ma_{Ex} \\
y:&\quad -B_y - mg = ma_{Ey} \\
z:&\quad F \cdot \frac{L}{2} + B_x \cdot \frac{L}{2} = I_E \alpha_2
\end{aligned}
$$

Then, we bridge from C because we know everything about that.

$$
\begin{aligned}
\vec{a}_D &= \cancelto{0}{\vec{a}_C} + \vec{\alpha}_1 + \vec{r}_{D/C} - \cancelto{0}{\omega^2 \vec{r}_{D/C}} \\
&= \alpha_1 \hat{k} \times \left(-\frac{L}{2}\right)\hat{j} \\
&= \left(\alpha_1 \frac{L}{2}\right)\hat{i} \\
\vec{a}_B &= \cancelto{0}{\vec{a}_C} + \alpha_1 \times \vec{r}_{B/C} - \cancelto{0}{\omega_1^2 \vec{r}_{B/C}} \\
\vec{a}_E &= \vec{a}_B + \vec{\alpha}_2 \times \vec{r}_{E/B} - \cancelto{0}{\omega_2^2 \vec{r}_{E/B}} \\
&= \alpha_1 \times \vec{r}_{B/C} + \vec{\alpha}_2 \times \vec{r}_{E/B} \\
&= \alpha_1 \hat{k} \times L\hat{j} + \alpha_2 \hat{k} \times \left(-\frac{L}{2}\right)\hat{j} \\
&= \left(\alpha_1 L + \frac{\alpha_2 L}{2}\right)\hat{i}
\end{aligned}
$$

Which gives us the remainder of the equations. Note that the y components of both acceleration is zero at that instant.

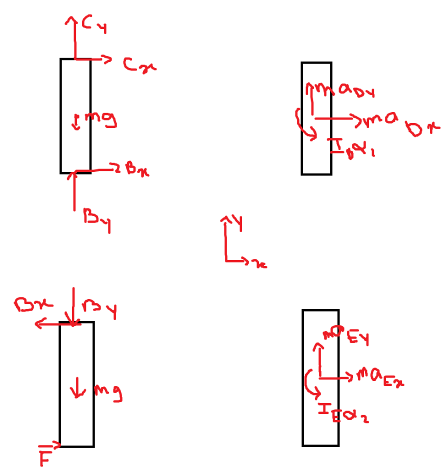

Figure 83: Example 65 FBD MAD

5/05: Rigid Body Work Energy

The equations for rigid-body work is:

$$W = \Delta E_K + \Delta E_P + \Delta E_S \tag{84}$$

$$= \int \vec{F} \cdot \mathrm{d}\vec{r} + \int \vec{M} \cdot \mathrm{d}\vec{\theta} \tag{85}$$

$$\Delta E_k = \frac{1}{2} m \left(v_{Gf}^2 - v_{G0}^2\right) + \frac{1}{2} I_G \left(\omega_f^2 - \omega_0^2\right) \tag{86}$$

$$\Delta E_P = mg \left(y_{Gf} - y_{G0}\right) \tag{87}$$

$$\Delta E_S = \frac{1}{2} k \left(s_f^2 - s_0^2\right) \tag{88}$$

where s is the displacement of a spring from its free length.

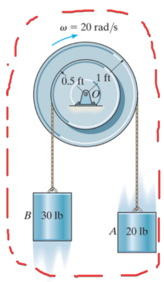

Figure 84: Example 67

Example 67

The double pulley consists of two separate pulleys that are welded together. It has a combined weight of 50lbs and a radius of gyration of 0.6ft about O. If it is initially turning with an angular velocity of 20rad/s clockwise, determine the angular velocity of the double pulley the instant the 10lb weight moves 2ft downward.

We are given:

$$m_O = \frac{50}{32.2} \text{ slugs}$$
$$m_A = \frac{20}{32.2} \text{ slugs}$$
$$m_B = \frac{30}{32.2} \text{ slugs}$$
$$r = 0.5 \text{ ft}$$
$$R = 1 \text{ ft}$$
$$k_0 = 0.6 \text{ ft}$$
$$I_0 = m_0 k_0^2$$
$$L_A = 2 \text{ ft}$$
$$\omega = 20 \text{ rad/s}$$

We will draw a dotted line to show the control volume. Anything that crosses it is work. Nothing is moving the boundary, so our work is 0. We have no springs so spring energy is also zero.

$$\cancelto{0}{W} = \Delta E_K + \Delta E_P + \cancelto{0}{\Delta E_S}$$

$$\Delta E_K = \frac{1}{2} m_A \left(v_{Af}^2 - v_{A0}^2\right) + \frac{1}{2} m_B \left(v_{Af}^2 - v_{A0}^2\right) \ldots$$

$$\cdots + \frac{1}{2} m_O \left(\cancelto{0}{v_{Of}^2} - \cancelto{0}{v_{O0}^2}\right) + \frac{1}{2} I_G \left(\omega_f^2 - \omega_0^2\right)$$

$$\Delta E_P = -m_A g L_A + m_B g L_B$$

We can cross out kinetic (linear) energy of the spinny thing because it is pinned. Linear speed at a point on the spinny thing is

$$v_{A0} = R\omega_0$$
$$v_{Af} = R\omega_f$$
$$v_{B0} = r\omega_0$$
$$v_{Bf} = r\omega_f$$

The angle of twist is the same for both. Using figure 85, we get another equation:

$$L_B = L_A \cdot \frac{r}{R}$$

and solving yields

$$\omega_f = \pm 20.4 \text{ rad/s}$$

but realistically, it would actually be minus because it is rotating counterclockwise.

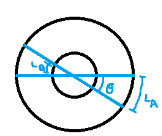

Figure 85: Example 67 Determining Length Relationship

Example 68

If the uniform 30kg slender rod starts from rest at the position shown. Determine its angular velocity after it as made 4 revolutions. The forces remain perpendicular to the rod.

As this rod turns, the forces also turn. Since this is pinned, displacement is zero. Let $F_1 = 30\,\text{N}, F_2 = 20\,\text{N}$, and the total length is $L = 3\,\text{m}$. The applied moment is clockwise with $M_O = 20\,\text{N}\cdot\text{m}$. We are asked to find ω_f. There are external forces, hence there is work. We have kinetic energy. Since the center of gravity begins and finishes at the same position, we do not have a change in potential energy. We do not have a spring. If a partial revolution was made, we may have had potential energy.

$$W = \Delta E_k + \cancelto{0}{\Delta E_P} + \cancel{\Delta E_S}$$

$$W = F_1(4)(2\pi \frac{L}{6}) + F_2(4)$$

$$\Delta E_K = \frac{1}{2}m\left(v_f^2 - \cancelto{0}{v_0^2}\right) + \frac{1}{2}$$

$$8\pi\left(F_1\cdot\frac{L}{6} + F_2\cdot\frac{L}{2} + M_O\right) = \frac{1}{2}mv_{Gf}^2 + \frac{1}{2}I_G\omega_f^2$$

Since all of them have a 8π term, we can take that out, for work. We have two unknowns: ω_f, v_{Gf}. We do not use bridge, because work-energy is scalar. We should not need to use parallel axis theorem in this class. We realize that the angular velocity and linear velocity is related, which brings the equation:

$$v_{Gf} = \omega_f \frac{L}{6}$$

Our ω_f will yield positive and negative. Since we took clockwise to be positive, it will be positive. However, we can keep it as positive and negative.

Just because something has a spring does not mean that it is work energy.

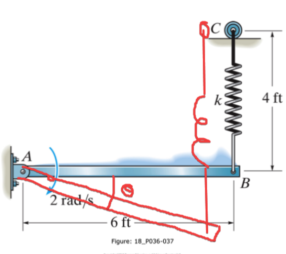

Figure 86: Example 70

Example 70

At the instant shown, the 50lb bar rotates clockwise at 2 rad/s. The spring attached to its end always remains vertical due to the massless roller at C. If the spring has an unstretched length of 2ft and a stiffness of 12lb/ft, determine the angle measured from the horizontal to which the bar rotates before it momentarily stops.

We are given

$$m = \frac{50}{32.2}\,\text{slugs}$$
$$L = 6\,\text{ft}$$
$$H = 4\,\text{ft}$$
$$\omega_0 = 2\,\text{rad/s}$$
$$L_{free} = 2\,\text{ft}$$
$$k = 12\,\text{lb/ft}$$

There is no *applied* forces, hence work is zero.

$$\cancelto{0}{W} = \Delta E_K + \Delta E_P + \Delta E_S$$

$$\Delta E_K = \frac{1}{2}m\left(\cancelto{0}{v_{Gf}^2} - v_{G0}^2\right) + \frac{1}{2}I_G\left(\cancelto{0}{\omega_f^2} - \omega_0^2\right)$$

$$\Delta E_P = mg\frac{L}{2}\sin\theta$$

$$\Delta E_S = \frac{1}{2}k\left(s_f^2 - s_0^2\right)$$

$$s_f = (H + L\sin\theta) - L_{free}$$
$$s_0 = H - L_{free}$$
$$0 = -\frac{1}{2}mv_{G0}^2 - \frac{1}{2}I_G\omega_0^2 + mg\frac{L}{2}\sin\theta\ldots$$
$$\cdots + \frac{1}{2}k\left[((H + L\sin\theta) - L_{free})^2 - (H - L_{free})^2\right]$$
$$v_{G0} = \omega_0\cdot\frac{L}{2}$$

We need MATLAB to solve this.

Example 71

The 20kg rod is released from rest when q=0°. Determine it's angular velocity when q=90°. The spring has an unstretched length of 0.5m

We are given $m = 20\,\text{kg}$ and $L_{free} = 0.5\,\text{m}$. It's a work-energy because there is a big dis-

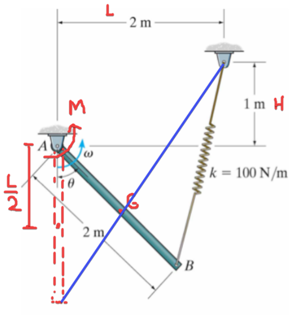

Figure 87: Example 71

placement. $L = 2\,\text{m}, H = 1\,\text{m}$. From the inertia sheet, $I_G = \frac{mL^2}{12}$. A professor addition to the problem: let there also be a $M_A = 20\,\text{N}\cdot\text{m}$. $k = 100\,\text{N/m}$ We have to find ω_f. Our work energy equation has all the terms:

$$W = \Delta E_K + \Delta E_P + \Delta E_S$$

$$W = M_A \cdot \frac{90\pi}{180}$$

$$\Delta E_K = \frac{1}{2}m\left(v_{Gf}^2 - \cancelto{0}{v_{G0}^2}\right) + \frac{1}{2}I_G\left(\omega_f^2 \cancelto{0}{\omega_0^2}\right)$$

$$v_{Gf} = \omega_f \frac{L}{2}$$

$$\Delta E_P = mg\frac{L}{2}$$

$$\Delta E_S = \frac{1}{2}k\left(s_f^2 - s_0^2\right)$$

$$s_0 = \sqrt{L^2 + (L+H)^2} - L_{free}$$

$$s_f = H - L_{free}$$

and then we substitute all that massive equation

$$M_A \cdot \frac{90\pi}{180} = \frac{1}{2}mv_{Gf}^2 + \frac{1}{2}I_G\omega_f^2 + \omega_f\frac{L}{2} + mg\frac{L}{2}\ldots$$

$$\cdots + \frac{1}{2}k\left[(H - L_{free})^2 - \left(\sqrt{L^2 + (L + H\right.\right.$$

The above equation has only one unknown, so we can solve them.

We only covered two topics, hence the exam will have two problems only: Newton's Second Law, and Work-Energy problem. We will also have to figure out what is going on. We can *potentially* have kinematics bridges on both problems, but they are more likely to be on the Newton's second law problem. We should *not* do the course evaluation. We will do them in class on Wednesday.

5/10: More problems

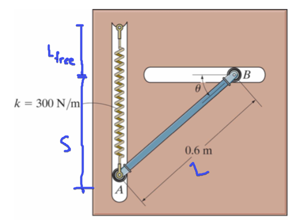

Figure 88: Example 73

Example 73

The spring is attached to the end of a 15kg rod and is unstretched when $\theta = 0°$. If the rod is released from rest, determine its angular velocity the instant $\theta = 30°$.

We are given $m, L, k, w_0 = 0$. The work energy is:

$$\cancelto{0}{W} = \Delta E_k + \Delta E_p + \Delta E_s$$

$$\Delta E_k = \frac{1}{2}m\left(v_{Gf}^2 - \cancelto{0}{v_{G0}^2}\right) + \frac{1}{2}I_0\left(\omega_f^2 - \cancelto{0}{\omega_0^2}\right)$$

$$\Delta E_p = -mg\frac{L}{2}\sin\theta$$

$$\Delta E_s = \frac{1}{2}k\left(s_f^2 - \cancelto{0}{s_0^2}\right)$$

$$s_f = L\sin\theta$$

$$0 = \frac{1}{2}mv_{Gf}^2 + \frac{1}{2}\frac{mL^2}{12}\omega_f^2 - mg\frac{L}{2}\sin\theta + \frac{1}{2}k(L\sin\theta^2)$$

We have to bridge, but the bridging will not be done using pins. The kinematics are:

$$\vec{v}_{Gf} = \vec{v}_{Af} + \vec{\omega}_f \times \vec{r}_{G/A}$$

$$= v_{Af}\hat{j} + \omega_f\hat{k} \times \left[\left(\frac{L}{2}\cos\theta\right)\hat{i}\left(\frac{L}{2}\sin\theta\right)\hat{j}\right]$$

$$= \left(-\omega_f\frac{L}{2}\sin\theta\right)\hat{i} + \left(v_{Af} + \omega_f\frac{L}{2}\cos\theta\right)\hat{j}$$

$$v_{Gf} = \sqrt{\left(-\omega_f\frac{L}{2}\sin\theta\right)^2 + \left(v_{Af} + \omega_f\frac{L}{2}\cos\theta\right)^2}$$

If both of the ends are moving, and we use one of the ends, we get one more equation, and more unknown. We should bridge across the whole thing:

$$\vec{v}_{Bf} = \vec{v}_{Af} + \vec{\omega}_f \times \vec{r}_{B/A}$$
$$v_{Bf}\hat{i} = v_{Af}\hat{j} + \omega_f\hat{k} \times \left[(L\cos\theta)\hat{i} + (L\sin\theta)\hat{j}\right]$$
$$i : v_{Bf} = -\omega_f L \sin\theta$$
$$j : 0 = v_{Af} + \omega_f L \cos\theta$$

We do the inertia and calculations about the center of gravity because otherwise we will have to deal with kinetic motions off centered. We have now all the tools needed to solve dynamics problem. This class shifts into vibrations.

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

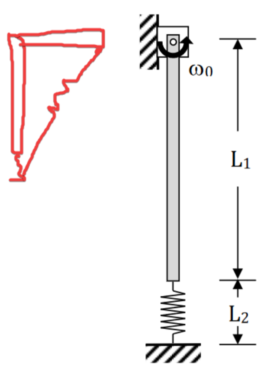

Figure 89: Example 72

Example 72

A pinned bar of mass of m has an initial angular velocity of ω_0 as shown. The bottom of the bar is attached to an unstretched spring with a stiffness k and an unstretched length of L_{free}

1. Find a single closed equation that could be used to find the angular velocity when the bar has rotated °
2. Find a closed set of equations that could be used to find the forces in the pin when the bar has rotated °

Part A is a work energy problem because it is swinging from vertical to horizontal. There's a moment at the pin. We are given $m, L_1, L_2, L_{free}, \omega_0, M_0$

$$W = \Delta E_K + \Delta E_P + \Delta E_S$$
$$W = M_0 \cdot \frac{90°\pi}{180°}$$
$$\Delta E_K = \frac{1}{2}m\left(v_{Gf}^2 - v_{G0}^2\right) + \frac{1}{2}I_G\left(\omega_f^2 - \omega_0^2\right)$$
$$v_{Gf} = \frac{L_1}{2}\omega_f$$
$$v_{G0} = \frac{L_1}{2}\omega_0$$
$$\Delta E_P = mg\frac{L_1}{2}$$
$$\Delta E_S = \frac{1}{2}k\left(s_f^2 - s_0^2\right)$$
$$s_f = L_f - L_{free}$$
$$L_f = \sqrt{L_1^2 + (L_1 + L_2)^2}$$
$$s_0 = L_0 - L_{free}$$

and now the super big combination of that equation

$$M_0\frac{\pi}{2} = \frac{1}{2}m\left((\frac{L_1}{2}\omega_f)^2 - ()\frac{L_1}{2}\omega_0)^2\right) + \frac{1}{2}I_G\left(\omega_f^2 + \omega_0^2\right)\dots$$
$$\dots + mg\frac{L_1}{2}\dots$$
$$\dots + \frac{1}{2}k\left((sqrtL_1^2 + (L_1 + L_2)^2 - L_{free})^2 - (L_2 - L_{free})^2\right)$$

Solving this will yield ω_f, the only unknown. Next, we do the forces. Using the FBD and MAD, we get:

$$x : \quad O_x - F_S\cos\phi = ma_{Gx}$$
$$y : \quad O_y - mg - F_S\sin\phi = ma_{Gy}$$
$$z : \quad -O_y\left(\frac{L_1}{2}\right) - F_S\sin\phi\left(\frac{L_1}{2}\right) + M_0 = I_G\alpha$$
$$F_S = ks_f$$
$$\phi = \arctan\left(\frac{L_1 + L_2}{L_1}\right)$$
$$a_{Gx} = -\omega_f^2\frac{L_1}{2}$$
$$a_{Gy} = \alpha\frac{L_1}{2}$$

Our missing stuff come from friction and springs. Additionally, for the acceleration, we bridge:

$$\begin{aligned}
\vec{a}_G &= \cancelto{0}{\vec{a}_0} + \vec{\alpha} \times \vec{r}_{G/O} - \omega_f^2 \vec{r}_{G/O} \\
&= \alpha \hat{k} \times \frac{L_1}{2}\hat{i} - \omega_f^2 \frac{L_1}{L_2}\hat{i} \\
&= \left(\frac{\alpha L_1}{2}\right)\hat{j} + \left(-\omega_f^2 \frac{L_1}{2}\right)\hat{i}
\end{aligned}$$

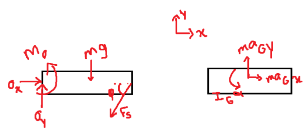

Figure 90: Example 72 FBD MAD