

# Notes

## MATH 152

Turibius Rozario

September 8, 2021

### Non-Class Info

Michael Romeo is the SI PASS leader, and has been doing it for a bit of time. Has two 50 minute sessions each week: one in-person, and other online. Specifically dedicated to CALC 2. Like an organized tutoring student. Will have review session prior to each exam.

Studies showed that: Those who attended SI PASS session perform better than those that do not. Michael will keep reminding us of this.

### 6.2 Volumes

The “ $dx$ ” represents the super small thickness. In general, formula for volume is:

$$V = \int_a^b A(x) dx$$

Last time we talked about volumes by the disk method, where if the cross-sections are circular disks, then:

$$A(x) \text{ or } A(y) = \pi r^2$$

Washer method is where, if the cross sections are washers, then:

$$A(x) \text{ or } A(y) = \pi [\text{outer}^2 - \text{inner}^2]$$

See Figure for view. The above are solids of revolutions. The greatest issue is in finding what formula to use for the radius for the area.

#### 0.1 Exercise

Finding volume of solid by revolving region by the bounded curves about the given  $y$ :

$$\begin{aligned} y &= \sqrt{x-1} \\ x &= 2 \\ x &= 5 \\ y &= 1 \end{aligned}$$

Fig 1: Washer

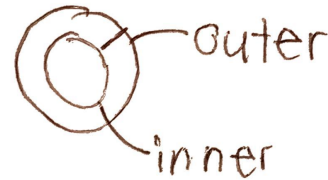
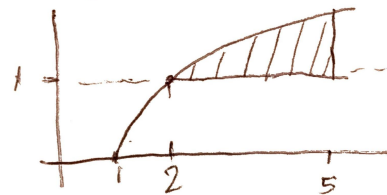


Fig 2: Exercise 1



This can be viewed in figure 0.1. Since it is being revolved around the  $x$ -axis, the revolution will be in respect to  $x$ . We will be generating a washer.

$$\text{outer} = y = \sqrt{x-1}$$

$$\text{inner} = y = 1$$

$$\begin{aligned} \therefore A(x) &= \pi [(\sqrt{x-1})^2 - 1] \\ &= \pi (x-2) \end{aligned}$$

Next, we find the volume.

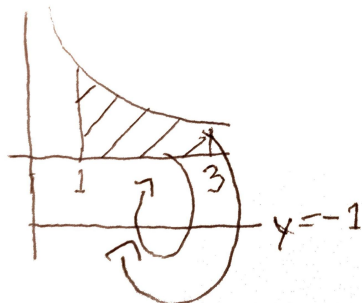
$$\begin{aligned} V &= \pi \int_2^5 (x-2) dx \\ &= \pi \left[ \frac{1}{2}x^2 - 2x \right]_2^5 \\ &= \pi \left[ \frac{25}{2} - 10 - (2-4) \right] \\ &= \frac{9}{2}\pi \end{aligned}$$

#### Exercise

For this exercise, the solid will be made by revolving around the  $y = -1$ . Visible in figure 0.1

$$\begin{aligned} y &= \frac{1}{x} \\ y &= 0 \\ x &= 1 \\ x &= 3 \end{aligned}$$

Fig 3: Exercise 2



Here, the radius are different.

$$\text{outer} = \frac{1}{x} + 1$$

$$\text{inner} = 1$$

$$A(x) = \pi \left[ \left( \frac{1}{x} + 1 \right)^2 - 1 \right]$$

$$= \pi (x^{-2} + 2x^{-1})$$

$$V(x) = \pi \int_1^3 (x^{-2} + 2x^{-1}) dx$$

$$= \pi \left[ -\frac{1}{x} + 2 \ln |x| \right]_1^3$$

$$= \pi \left( -\frac{1}{3} + \ln 9 - (-1) \right)$$

$$= \pi \left( \frac{2}{3} + \ln 9 \right)$$

## 6.3 Cylindrical Shells Method

For regions made by a single curve, or where the normal method would not work, we can use the cylindrical shells method.

### 0.2 Exercise

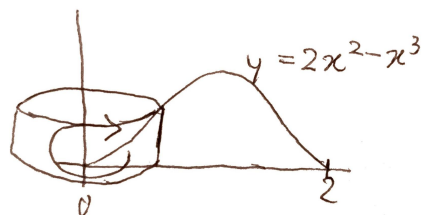
For example for this:

$$f(x) = 2x^2 - x^3$$

$$y = 0$$

If revolved around the y-axis, it should resemble Figure 0.2. It would not be impossible, but certainly difficult. This is because we would have to solve it in terms of y. This is not an issue for most other curves, but for this, is.

Fig 4: Exercise 3



Hence, the cylindrical method can be used, according to which:

$$V(x) = \int_a^b [\text{circumference}] \times [\text{height}] dx$$

Where  $dx$  is the thickness. For this problem, it would resemble this:

$$V(x) = \int_0^2 2\pi \cdot x \cdot f(x) dx$$

$$= 2\pi \int_0^2 x(2x^2 - x^3) dx$$

$$= 2\pi \int_0^2 2x^3 - x^4 dx$$

$$= 2\pi \left[ \frac{1}{2}x^4 - \frac{1}{5}x^5 \right]_0^2$$

$$= 2\pi \left[ 8 - \frac{32}{5} - (0 - 0) \right]$$

$$= 2\pi \times \frac{8}{5}$$

$$= \frac{16}{5}\pi$$

Be aware of what the question allows you to do. In some cases, you can use both, in others, only one.

### 0.3 Exercise

We are given:

$$x = \sqrt{y}, x = 0, y = 1$$

And this will be revolved about the x-axis to form a solid. The figure can be seen in Figure 0.3. Since this problem is about the x-axis, the problem will be done about the y. Since the cylinder is leaning on the side, the height will be the  $x = \sqrt{y}$ , and the radius would be the y:

Fig 5: Exercise 4

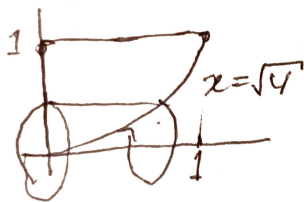
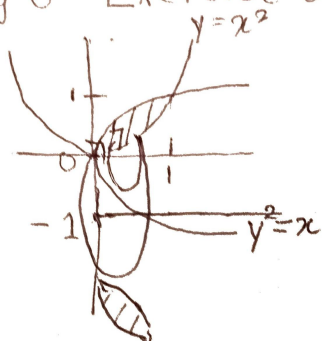


Fig 7: Exer Avg Val



Fig 6: Exercise 5



$$\begin{aligned} V(x) &= 2\pi \int_0^1 y^{\frac{3}{2}} dy \\ &= 2\pi \left[ \frac{2}{5} y^{\frac{5}{2}} \right]_0^1 \\ &= 2\pi \cdot \frac{2}{5} \\ &= \frac{4}{5}\pi \end{aligned}$$

#### 0.4 Exercise

Similarly, this can be used for different axis. If given

$$y = x^2, x = y^2$$

, about the  $y = -1$ . See Figure 0.4. Here:

$$\begin{aligned} \text{For each } y \in [0, 1] \\ \text{radius} &= y - (-1) = y + 1 \\ \text{height} &= \sqrt{y} - y^2 \end{aligned}$$

Now, apply the formula in terms of  $y$ :

$$\begin{aligned} V &= 2\pi \int_0^1 (y+1)(\sqrt{y}-y^2) dy \\ &= 2\pi \int_0^1 (y^{\frac{3}{2}} - y^3 + \sqrt{y} - y^2) \\ &= 2\pi \left[ \frac{2}{5} y^{\frac{5}{2}} - \frac{1}{4} y^4 + \frac{2}{3} y^{\frac{3}{2}} - \frac{1}{3} y^3 \right]_0^1 \\ &= 2\pi \left[ \frac{2}{5} - \frac{1}{4} + \frac{2}{3} - \frac{1}{3} \right] \\ &= 2\pi \left( \frac{24 - 15 + 20}{60} \right) \\ &= 2\pi \times \frac{29}{60} \\ &= \frac{29}{30}\pi \end{aligned}$$

### 6.5 Finding Average Value of a Function

If we let  $f(x)$  to be a continuous function on  $[a, b]$ , then the average value of  $f$  can be found by the integral. If we pretend that this class has 100 students, then we add up all our test scores, and divide by the number of students, to gain the average score. This is for finite. However, for a continuous function, we have to use an integral, and then divide by the size of the domain (i.e.  $b - a$ ).

For the ease of understanding, assume  $f(x) \geq 0, \forall x \in [a, b]$ . This can be used for negatives too. For example, in Figure 0.4, the average would be a rectangle such that the height of the rectangle is a constant function, such that the area under that constant function is equal to the area of  $f(x)$ .

Definition:  $f_{avg}$  is the  $y$ -value such that the rectangle with base  $[a, b]$  and height  $f_{avg}$  has the same area as the region under the graph of  $f$ .

Since it is a rectangle, the area is geometric:

$$A = f_{avg}(b - a) = \int_a^b f(x) dx$$

Consecuntely:

$$f_{avg} = \frac{\int_a^b f(x) dx}{b - a}$$

This relates to the Mean Value Theroem (MVT). According to this theorem, if  $f(x)$  is a continuous function, at least once will it achieve its average value.

Definition: If  $f$  is continuous on  $[a, b]$ , then:

$$\exists c \in [a, b] \ni f(c) = f_{avg}$$

## 0.5 Exercise

Finding the average value of the function on the given interval:

$$f(x) = \frac{1}{x}, x \in [1, 4]$$

$$\begin{aligned} f_{avg} &= \frac{1}{4 - 1} \int_1^4 \frac{1}{x} dx \\ &= \frac{1}{3} [\ln |x|]_1^4 \\ &= \frac{1}{3} \ln 4 \end{aligned}$$

## 0.6 Exercise

Finding the average value for the following:

$$g(x) = x^2 \sqrt{1 + x^3}, x \in [0, 2]$$

$$\begin{aligned} g_{avg} &= \frac{1}{2 - 0} \int_0^2 x^2 \sqrt{1 + x^3} dx \\ \text{Let } u &= 1 + x^3 \\ \Rightarrow x^2 dx &= \frac{1}{3} du \\ g_{avg} &= \frac{1}{2} \times \frac{1}{3} \int_1^9 \sqrt{u} du \\ &= \frac{1}{6} \left[ \frac{2}{3} u^{\frac{3}{2}} \right]_1^9 \\ &= \frac{1}{9} (27 - 1) \\ &= \frac{26}{9} \end{aligned}$$

# Notes

## MATH 152

Turibius Rozario

September 13, 2021

### Section 6.2 Continued

Last time the average value of  $f$  was discussed. Accordingly:

If  $f(x)$  is continuous on  $[a, b]$ , then the average value of  $f$  on  $[a, b]$  is:

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x) dx$$

And according to the **MVT of Integrals**: If  $f$  is continuous on  $[a, b]$ , then there exists at least one  $c \in [a, b] \ni f(c) = f_{avg}$

#### Exercise 1

Finding the average of the function on the given interval:

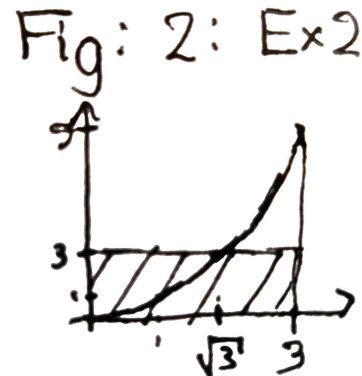
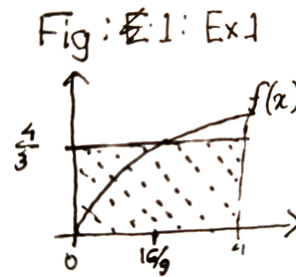
$$f(x) = \sqrt{x}, [0, 4]$$

$$\begin{aligned} f_{avg} &= \frac{1}{4-0} \int_0^4 \sqrt{x} dx \\ &= \frac{1}{4} \times \frac{2}{3} \left[ x^{\frac{3}{2}} \right]_0^4 \\ &= \frac{1}{6} (8-0) \\ &= \frac{4}{3} \end{aligned}$$

Finding all the  $c$  in the interval where  $f(c) = f_{avg}$ .

$$\begin{aligned} f(c) &= \frac{4}{3} \\ \sqrt{c} &= \frac{4}{3} \\ c &= \frac{16}{9} \end{aligned}$$

Sketch of the graph of  $f$  and the rectangle that has same area is in Fig



#### Exercise 2

For

$$f(x) = x^2, x \in [0, 3]$$

The average value is:

$$\begin{aligned} f_{avg} &= \frac{1}{3-0} \int_0^3 x^2 dx \\ &= \frac{1}{9} \left[ x^3 \right]_0^3 \\ &= \frac{1}{9} \times 27 \\ &= 3 \end{aligned}$$

To find the point  $c$ :

$$\begin{aligned} f(c) &= 3 \\ c^2 &= 3 \\ c &= \sqrt{3} \end{aligned}$$

The area of the rectangle can be seen in Fig

## 7.1 Integration by Parts

The end goal is not only to know the integrals, but also to be just able to look at an integral, and immediately being able to recognize it. We have to be able to recognize it too. Integration by parts stems from the product rule:

$$\begin{aligned}\frac{d}{dx}[f(x)g(x)] &= f(x)g'(x) + g(x)f'(x) \\ \therefore \int [f(x)g'(x) + g(x)f'(x)] dx &= f(x)g(x) \\ \int f(x)g'(x) dx + \int g(x)f'(x) dx &= f(x)g(x) \\ \implies \int f(x)g'(x) dx &= f(x)g(x) - \int g(x)f'(x) dx \\ \therefore \int u dv &= uv - \int v du\end{aligned}$$

Where:

$$\begin{aligned}u &= f(x) \\ du &= f'(x) \\ dv &= g'(x) dx \\ v &= g(x)\end{aligned}$$

This should be used and is useful when the integral on the right hand side (RHS) is easier than the one on the left hand side (LHS). For definite integrals, it is:

$$\int_a^b f(x)g'(x) dx = f(x)g(x)|_a^b - \int_a^b g(x)f'(x) dx$$

These are used generally when we are integrating the product of two different kinds of function, such as polynomial and exponential.

### Exercise 3

$$\begin{aligned}\int xe^{-x} dx \\ u = x \\ du = dx \\ dv = e^{-x} dx \\ v = -e^{-x} \\ \int u dv = uv - \int v du \\ = -xe^{-x} + \int e^{-x} dx \\ = -xe^{-x} - e^{-x} + C\end{aligned}$$

If the  $u$  and the  $dv$  were switched, it would create issues:

$$u = e^{-x}; dv = x dx \implies du = -e^{-x} dx; v = \frac{1}{2}x^2$$

$$\int xe^{-x} dx = \frac{1}{2}x^2e^{-x} + \int \frac{1}{2}x^2e^{-x}$$

The result, by no means, is easy, hence, although right, not the best.

Mnemonic to memorize the order: **L**ogarithms  
**I**nverse Trigonometry  
**P**olynomial  
**E**xponential  
**T**rigonometry

### Exercise 4

Given the following integral, evaluate.

$$\int p^5 \ln p dp$$

Using LIPET, we know that:

$$u = \ln p; dv = p^5 dp \implies du = \frac{1}{p} dp; v = \frac{1}{6}p^6$$

$$\begin{aligned}\int p^5 \ln p dp &= uv - \int v du \\ &= \frac{1}{6}p^6 \ln p - \int \frac{1}{6}p^5 dp \\ &= \frac{1}{6}p^6 \ln p - \frac{1}{36}p^6 + C\end{aligned}$$

### Exercise 5

$$\int t^3 e^t dt = t^3; dv = e^t dt \implies du = 3t^2 dt; v = e^t$$

$$\int t^3 e^t dt = t^3 e^t - \int 3t^2 e^t dt$$

Here, we have to do Integrals by Parts again for the latter integral:

$$u = 3t^2; dv = e^t dt \implies du = 6t dt; v = e^t$$

Now, continuing:

$$\begin{aligned}&= t^3 e^t - [3t^2 e^t - \int 6te^t dt] \\ n &= t^3 e^t - 3t^2 e^t + \int 6te^t dt\end{aligned}$$

And yet again:

$$\begin{aligned} u &= 6t; dv = e^t dt du = 6 dt; v = e^t \\ &= t^3 e^t - 3t^2 e^t + 6te^t - \int 6e^t dt \\ &= t^3 e^t - 3t^2 e^t + 6te^t - 6e^t + C \end{aligned}$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \csc^2 x dx = -x \cot x \Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x} dx$$

$$u = \sin x$$

$$du = \cos x dx$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \csc^2 x dx = [-x \cot x + \ln |\sin x|]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= -\frac{\pi}{2}(0) + \ln 1 - \left[ -\frac{\pi}{4} + \ln \frac{\sqrt{2}}{2} \right]$$

$$= \frac{\pi}{4} - \ln \frac{\sqrt{2}}{2}$$

### Exercise 6

$$\int \sin^{-1} x dx = \sin^{-1} x; dv = dx du = \frac{1}{\sqrt{1-x^2}} dx; v = x$$

$$\int \sin^{-1} x dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} dx$$

$$u = 1 - x^2$$

$$du = -2x dx$$

$$\begin{aligned} \therefore \int \sin^{-1} x dx &= x \sin^{-1} x + \frac{1}{2} \int \frac{1}{\sqrt{u}} du \\ &= x \sin^{-1} x + \frac{1}{2} \times 2u^{\frac{1}{2}} + C \\ &= x \sin^{-1} x + \sqrt{1-x^2} + C \end{aligned}$$

### Exercise 7

The following is an exponential function with a trigonometry function.

$$\int e^{-x} \cos x dx = e^{-x}; dv = \cos x dx du = -e^{-x} dx; v = \sin x$$

$$\int e^{-x} \cos x dx = e^{-x} \sin x + \int e^{-x} \sin x dx$$

$$u = e^{-x}$$

$$dv = \sin x dx$$

$$du = -e^{-x} dx$$

$$v = -\cos x$$

$$\int e^{-x} \cos x dx = e^{-x} \sin x - e^{-x} \cos x - \int e^{-x} \cos x dx$$

$$2 \int e^{-x} \cos x dx = e^{-x} \sin x - e^{-x} \cos x$$

$$\int e^{-x} \cos x dx = \frac{e^{-x}}{2} (\sin x - \cos x) + C$$

### Exercise 8

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} x \csc^2 x dx = x; dv = \csc^2 x du = dx; v = -\cot x$$

## Discussion: Review 6.2

Usually, Wednesdays are for quiz, and we should give the TA 1 week. After integrating, remove the integrand. n

### 6.2 1

Finding the volume of the solid formed by rotating the region bounded by  $y = \sqrt{x}$ ,  $y = 2$ , and y-axis about the given line. Sketch the region, the solid, and a representative disk or washer.

1. The y-axis. **Answer:**

$$x = y^2$$

$$A = \pi(x)^2$$

$$V = \pi \int_0^2 (y^2)^2 dy$$

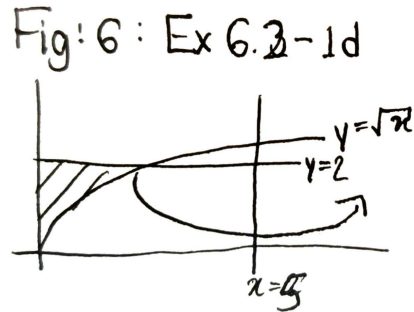
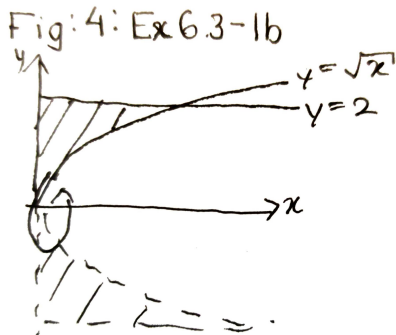
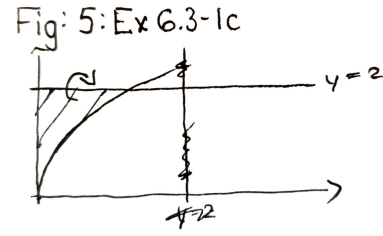
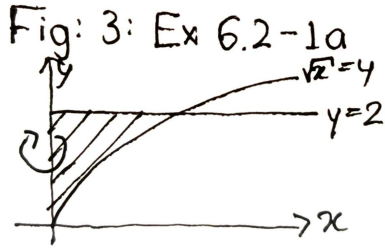
$$= \pi \frac{1}{5} [y^5]_0^2$$

$$= \frac{1}{5} \pi (32 - 0)$$

$$= \frac{1}{5} \times 32\pi$$

$$= \frac{32}{5} \pi$$

Diagram is shown at Figure 1



4.  $x = 5$ . Diagram is shown at Figure 4

2. The x-axis. **Answer:**

$$\begin{aligned} \sqrt{x} &= 2 \\ x &= 4 \\ A &= \pi(2^2 - x) \\ V &= \pi \int_0^4 4 - x \, dx \\ &= \pi \left[ 4x - \frac{1}{2}x^2 \right]_0^4 \\ &= \pi(16 - 8 - (0 - 0)) \\ &= 8\pi \end{aligned}$$

Diagram is shown at Figure 2

3.  $y = 2$

$$\begin{aligned} r &= (2 - \sqrt{x}) \\ V &= \pi \int_0^4 (2 - \sqrt{x})^2 \, dx \\ &= \pi \int_0^4 4 - 4\sqrt{x} + x \, dx \\ &= \pi \left[ 4x - \frac{4 \times 2}{3} x^{\frac{3}{2}} + \frac{x^2}{2} \right]_0^4 \\ &= \pi \left[ \left( 16 - \frac{8}{3} \times 8 + 8 \right) - (0 - 0 + 0) \right] \\ &= \frac{8}{3}\pi \end{aligned}$$

Diagram is shown at Figure 3

$$r_2 = 5$$

$$r_1 = 5 - y^2$$

$$\begin{aligned} A &= \pi(5^2 - (5 - y^2)^2) \\ &= \pi(25 - 25 + 10y^2 - y^4) \end{aligned}$$

$$\begin{aligned} V &= \pi \int_0^2 10y^2 - y^4 \, dy \\ &= \pi \left[ \frac{10}{3}y^3 - \frac{1}{5}y^5 \right]_0^2 \\ &= \pi \left( \frac{10 \times 8}{3} - \frac{32}{5} \right) \\ &= \frac{304}{15}\pi \end{aligned}$$

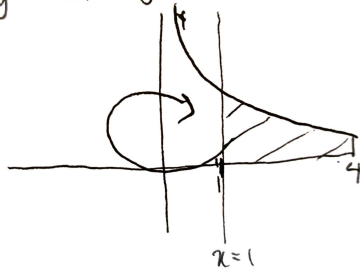
### 6.3 1

Given that

$$y = \frac{1}{x}, \quad x = 1, \quad x = 4$$

find the solid obtained by rotating the region bounded by above by y-axis, if it is above the x-axis. Use the cylindrical method. Diagram is shown in

Fig 7: Ex 6.3-1



Figure

$$\begin{aligned} V &= 2\pi \int_1^4 [\text{radius}] \times [\text{height}] dx \\ &= 2\pi \int_1^4 x \times \frac{1}{x} dx \\ &= 2\pi [x]_1^4 \\ &= 2\pi(4 - 1) \\ &= 6\pi \end{aligned}$$

# Notes

## MATH 152

Turibius Rozario

September 15, 2021

First half of class would be for sin and cos trigonometric integrals, and later sec and tan

### 7.2 Trigonometric Integrals

Strategy for evaluating integrals of the form:

$$\int \sin^m x \cos^n x dx \tag{1}$$

Where  $m, n$  are positive non-zero integers. If either powers are odd, it's easier.

1. If  $n$  is odd:  $n = 2k + 1$ :

$$\begin{aligned} \int \sin^m x \cos^{2k+1} x dx &= \int \sin^m x (\cos^2 x)^k \cos x dx \\ &= \int \sin^m x (1 - \sin^2 x)^k \cos x dx \end{aligned}$$

Then let  $u = \sin x$

This lets us cross out the the cos.

2. If  $m$  is odd:  $m = 2k + 1$ :

$$\begin{aligned} \int \sin^{2k+1} x \cos^n x dx &= \int (\sin^2 x)^k \cos^n x \sin x dx \\ &= \int (1 - \cos^2 x)^k \cos^n x \sin x dx \end{aligned}$$

Then let  $u = \cos x$

If both are odd, then either (a) or (b) can be used.

3. If  $m$  and  $n$  are both even, we have to use half-angle identities:

$$\sin^2 = \frac{1}{2}(1 - \cos 2x) \tag{2}$$

$$\cos^2 = \frac{1}{2}(1 + \cos 2x) \tag{3}$$

$$\sin x \cos x = \frac{1}{2} \sin 2x \tag{4}$$

### Example 1

$$\begin{aligned}\int \sin^6 x \cos^3 x \, dx &= \int \sin^6 x \cos^2 x \cos x \, dx \\ &= \int \sin^6 x (1 - \sin^2 x) \cos x \, dx \\ u &= \sin x \\ du &= \cos x \, dx \\ &= \int u^6 (1 - u^2) \, du \\ &= \int (u^6 - u^8) \, du \\ &= \frac{1}{7} u^7 - \frac{1}{9} u^9 + C \\ &= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C\end{aligned}$$

### Example 2

Will be a bit more difficult because the following has a 2 instead of a 1:

$$\begin{aligned}\int \sin^2 x \, dx &= \int \sin^4 2x \sin 2x \, dx \\ &= \int (1 - \cos^2 2x)^2 \sin 2x \, dx \\ u &= \cos 2x \\ du &= -2 \sin 2x \, dx \\ &= -\frac{1}{2} \int (1 - u^2)^2 \, du \\ &= -\frac{1}{2} \int (1 - 2u^2 + u^4) \, du \\ &= -\frac{1}{2} \left[ u - \frac{2}{3} u^3 + \frac{1}{5} u^5 \right] + C \\ &= -\frac{1}{2} \cos 2x + \frac{1}{3} \cos^3 2x - \frac{1}{10} \cos^5 2x + C\end{aligned}$$

### Example 3

Now, for evens:

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \cos^x dx &= \int_0^{\frac{\pi}{2}} \frac{1}{2}(1 + \cos 2x) dx \\
&= \frac{1}{2} \left( x + \frac{1}{2} \sin 2x \right)_0^{\frac{\pi}{2}} \\
&= \frac{1}{2} \left[ \frac{\pi}{2} + 0 - (0 + 0) \right] \\
&= \frac{\pi}{4}
\end{aligned}$$

The higher the even power, the more difficult it gets

#### Example 4

$$\begin{aligned}
\int_0^{\pi} \sin^4 3t dt &= \int_0^{\pi} (\sin^2 3t)^2 dt \\
&= \int_0^{\pi} \left[ \frac{1}{2}(1 - \cos 6t) \right]^2 dt \\
&= \frac{1}{4} \int_0^{\pi} (1 - 2 \cos 6t + \cos^2 6t) dt \\
&= \frac{1}{4} \int_0^{\pi} \left( 1 - 2 \cos 6t + \frac{1}{2}(1 + \cos 12t) \right) dt \\
&= \frac{1}{4} \int_0^{\pi} \left[ \frac{3}{2} - 2 \cos 6t + \frac{1}{2} \cos 12t \right] dt \\
&= \frac{1}{4} \left[ \frac{3}{2}t - \frac{1}{3} \sin 6t + \frac{1}{24} \sin 12t \right]_0^{\pi} \\
&= \frac{1}{4} \left[ \frac{3\pi}{2} - 0 + 0 - (0 - 0 + 0) \right] \\
&= \frac{3\pi}{8}
\end{aligned}$$

### Example 5

$$\begin{aligned}\int x \cos^2 x \, dx &= \frac{1}{2} \int x(1 + \cos 2x) \, dx \\ &= \frac{1}{2} \int (x + x \cos 2x) \, dx \text{ (Int. by parts)} \\ &= \frac{1}{4}x^2 + \frac{1}{2} \int x \cos 2x \, dx \\ u &= x \\ dv &= \cos 2x \, dx \\ du &= dx \\ v &= \frac{1}{2} \sin 2x \\ &= \frac{1}{4}x^2 + \frac{1}{2} \left[ \frac{1}{2}x \sin 2x - \int \frac{1}{2} \sin 2x \, dx \right] \\ &= \frac{1}{4}x^2 + \frac{1}{4}x \sin 2x - \frac{1}{4} \int \sin 2x \, dx \\ &= \frac{1}{4}x^2 + \frac{1}{4}x \sin 2x + \frac{1}{8} \cos 2x + C\end{aligned}$$

### Example 6

For this, we can either replace both of the trigonometry, or even better, just use the third formula in (c):

$$\begin{aligned}\int_0^{\frac{\pi}{2}} \sin^2 x \cos^2 x \, dx &= \int_0^{\frac{\pi}{2}} (\sin x \cos x)^2 \, dx \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{1}{2} \sin 2x\right)^2 \, dx \\ &= \frac{1}{4} \int_0^{\frac{\pi}{2}} \sin^2 2x \, dx \\ &= \frac{1}{8} \int_0^{\frac{\pi}{2}} (1 - \cos 4x) \, dx \\ &= \frac{1}{8} \left(x - \frac{1}{4} \sin 4x\right)_0^{\frac{\pi}{2}} \\ &= \frac{1}{8} \left[\frac{\pi}{2} - 0 - (0 - 0)\right] \\ &= \frac{\pi}{16}\end{aligned}$$

### Example 7

This doesn't look like the trig above, but it is:

$$\begin{aligned}
\int \cot^5 x \sin^4 x \, dx &= \int \frac{\cos^5 x}{\sin^5 x} \cdot \sin^4 x \, dx \\
&= \int \frac{\cos^5 x}{\sin x} \, dx \\
&= \int \frac{\cos^4 x}{\sin x} \cdot \cos x \, dx \\
&= \int \frac{(1 - \sin^2 x)^2}{\sin x} \cdot \cos x \, dx \\
u &= \sin x \\
du &= \cos x \, dx \\
&= \int \frac{(1 - u^2)^2}{u} \, du \\
&= \int \frac{1 - 2u^2 + u^4}{u} \, du \\
&= \int \left( \frac{1}{u} - 2u + u^3 \right) \, du \\
&= \ln |u| - u^2 + \frac{1}{4}u^4 + C \\
&= \ln |\sin x| - \sin^2 x + \frac{1}{4} \sin^4 x + C
\end{aligned}$$

## Using tan and sec in integrals

For the form:

$$\int \tan^m x \sec^n x \, dx \tag{5}$$

For this, keep the following in mind:

$$\sec^2 x - 1 = \tan^2 x \tag{6}$$

1. If  $n$  is even and at least 2:  $n = 2k, k \geq 1$ :

$$\begin{aligned}
\int \tan^m \sec^{2k} x \, dx &= \int \tan^m x (\sec^2 x)^{k-1} \sec^2 x \, dx \\
&= \int \tan^m x (1 + \tan^2 x)^{k-1} \sec^2 x \, dx
\end{aligned}$$

Then let  $u = \tan x$

2. If  $m$  is odd is odd  $m = 2k + 1$  and  $n \geq 1$ :

$$\begin{aligned}\int \tan^{2k+1} x \sec^n x \, dx &= \int (\tan^2 x)^k \sec^{n-1} x \sec x \tan x \, dx \\ &= \int (\sec^2 x - 1)^k \sec^{n-1} x \sec x \tan x \, dx\end{aligned}$$

Then let  $u = \sec x$

3. Other cases have no clear cut strategies.

For the following, these come up often, so we should memorize them:

1.

$$\begin{aligned}\int \tan x \, dx &= \int \frac{\sin x}{\cos x} \, dx \\ u &= \cos x \\ du &= -\sin x \, dx \\ &= -\int \frac{1}{u} \, du \\ &= -\ln |u| + C \\ &= -\ln |\cos x| + C \\ &= \ln |\sec x| + C\end{aligned}$$

Hence,  $\int \tan x \, dx = \ln |\sec x| + C$

2.

$$\begin{aligned}\int \sec x \, dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x} \, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x \tan x} \, dx \\ u &= \sec x + \tan x \\ du &= \sec x \tan x + \sec^2 x \\ &= \int \frac{1}{u} \, du \\ &= \ln |u| + C \\ \therefore \int \sec x \, dx &= \ln |\sec x + \tan x| + C\end{aligned}$$

3.

$$\int \sec^3 x \, dx$$

$$u = \sec x$$

$$dv = \sec^2 x \, dx$$

$$du = \sec x \tan x \, dx$$

$$v = \tan x$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$\int \sec^3 x \, dx = \frac{1}{2} [\sec x \tan x + \ln |\sec x + \tan x|] + C$$

### Examples of Type (a) and Type (b)

(a)

$$\begin{aligned} \int_0^{\frac{\pi}{4}} \sec^4 x \tan^4 x \, dx &= \int_0^{\frac{\pi}{4}} \sec^2 x \tan^4 x \sec^2 x \, dx \\ &= \int_0^{\frac{\pi}{4}} (1 + \tan^2 x) \tan^4 x \sec^2 x \, dx \end{aligned}$$

$$u = \tan x$$

$$du = \sec^2 x \, dx$$

$$= \int_0^1 (1 + u^2) u^4 \, du$$

$$= \int_0^1 (u^4 + u^6) \, du$$

$$= \left[ \frac{1}{5} u^5 + \frac{1}{7} u^7 \right]_0^1$$

$$= \frac{1}{5} + \frac{1}{7}$$

$$= \frac{12}{35}$$

(b)

$$\begin{aligned}\int \tan^3 2x \sec^5 2x \, dx &= \int \tan^2 2x \sec^4 2x \sec 2x \tan 2x \, dx \\ &= \int (\sec^2 2x - 1) \sec^4 2x \sec 2x \tan 2x \, dx \\ u &= \sec 2x \\ du &= 2 \sec 2x \tan 2x \, dx \\ &= \frac{1}{2} \int (u^2 - 1)u^4 \, du \\ &= \frac{1}{2} \int (u^6 - u^4) \, du \\ &= \frac{1}{14}u^7 - \frac{1}{10}u^5 + C \\ &= \frac{1}{14} \sec^7 2x - \frac{1}{10} \sec^5 2x + C\end{aligned}$$

(c)

$$\begin{aligned}\int \tan^2 x \sec x \, dx &= \int (\sec^2 x - 1) \sec x \, dx \\ &= \int \sec^3 x \, dx - \int \sec x \, dx \\ &= \frac{1}{2}(\sec x \tan x + \ln |\sec x + \tan x|) - \ln |\sec x + \tan x| + C \\ &= \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln |\sec x + \tan x| + C\end{aligned}$$

# Notes

## MATH 152

Turibius Rozario

September 20, 2021

Today we will be doing trig substitutions, 3 kinds in specific. The unit is 7.3. The particular substitution will depend on the pattern.

Integrals involving  $\sqrt{a^2 - x^2}$ ,  $a > 0$  will have  $x$  replaced with  $a \cdot \sin \theta$ . When it is a definite integral, the angle  $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ . This is because we need to restrain the sin function. The  $dx$  in the integral will be replaced with  $dx = a \cos \theta d\theta$ . Consequently:

$$\begin{aligned}\sqrt{a^2 - x^2} &= \sqrt{a^2 - a^2 \sin^2 \theta} \\ \sqrt{a^2 \cos^2 \theta} &= a |\cos \theta| \\ &= a \cos \theta\end{aligned}$$

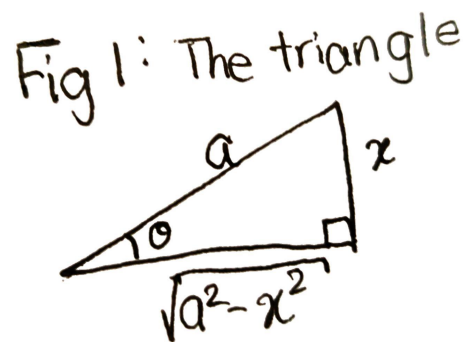
When doing indefinite integrals, we have to draw a triangle, so that we can replace the trigonometry with algebra involving  $x$ . Diagram of a triangle is shown in Figure .

For example:

$$\int_0^{2\sqrt{3}} \frac{x^3}{\sqrt{16 - x^2}} dx$$

To solve this:

$$\begin{aligned}x &= 4 \sin \theta \\ dx &= 4 \cos \theta d\theta\end{aligned}$$



This turns the limits of integration to:

$$\begin{aligned}
 &= \int_0^{\frac{\pi}{3}} \frac{64 \sin^3 \theta}{4 \cos \theta} \cdot 4 \cos \theta d\theta \\
 &= 64 \int_0^{\frac{\pi}{3}} \sin^3 \theta d\theta \\
 &= 64 \int_0^{\frac{\pi}{3}} (1 - \cos^2 \theta) \sin \theta d\theta \\
 &u = \cos \theta \\
 &du = -\sin \theta d\theta \\
 &= 64 \int_{\frac{1}{2}}^1 (1 - u^2) du \\
 &= 64 \left[ u - \frac{1}{3} u^3 \right]_{\frac{1}{2}}^1 \\
 &= 64 \left( \frac{5}{24} \right) \\
 &= \frac{40}{2}
 \end{aligned}$$

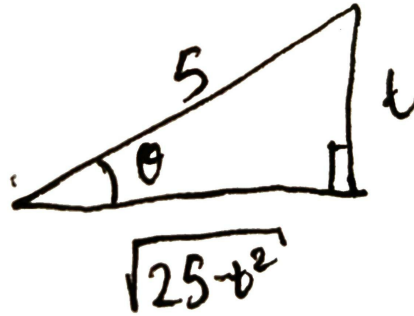
For indefinite integrals, the right triangle has to be used (note that this could have been done with u substitution)

$$\begin{aligned}
 &\int \frac{t}{\sqrt{25-t^2}} dt \\
 &t = 5 \sin \theta \\
 &dt = 5 \cos \theta d\theta \\
 &\int \frac{t}{\sqrt{25-t^2}} dt = \int \frac{5 \sin \theta}{\sqrt{25-25 \sin^2 \theta}} \cdot 5 \cos \theta d\theta \\
 &= \frac{5 \sin \theta \times 5 \cos \theta}{5 \cos \theta} d\theta \\
 &= \int 5 \sin \theta d\theta \\
 &= -5 \cos \theta + C
 \end{aligned}$$

However, since it originally had  $x$ , we have to switch it back using the triangle in Figure :

A more difficult example is:

Fig 2: Ex 2



$$\begin{aligned} \int \frac{1}{(5 - 4x - x^2)^{\frac{5}{2}}} dx &= \int \frac{1}{[5 - (x^2 + 4x)]^{\frac{5}{2}}} dx \\ &= \int \frac{1}{[9 - (x^2 + 4x + 4)]^{\frac{5}{2}}} dx \\ &= \int \frac{1}{[9 - (x + 2)^2]^{\frac{5}{2}}} dx \end{aligned}$$

$$x + 2 = 3 \sin \theta$$

$$dx = 3 \cos \theta d\theta$$

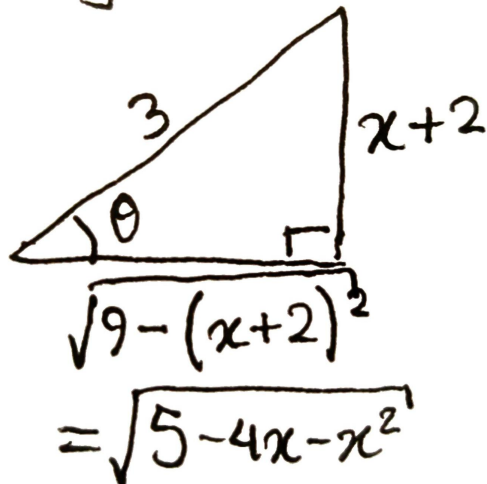
$$\begin{aligned} \int \frac{1}{(5 - 4x - x^2)^{\frac{5}{2}}} dx &= \int \frac{3 \cos \theta}{(3 \cos \theta)^5} d\theta \\ &= \frac{1}{81} \int \frac{1}{\sec^4 \theta} d\theta \\ &= \frac{1}{81} \int (1 + \tan^2 \theta) \sec^2 \theta d\theta \end{aligned}$$

$$u = \tan \theta$$

$$du = \sec^2 \theta d\theta$$

$$\begin{aligned} \int \frac{1}{(5 - 4x - x^2)^{\frac{5}{2}}} dx &= \frac{1}{81} \int (1 + u^2) du \\ &= \frac{1}{81} u + \frac{1}{243} u^3 + C \\ &= \frac{1}{81} \tan \theta + \frac{1}{243} \tan^3 \theta + C \\ &= \frac{1}{81} \left( \frac{x + 2}{\sqrt{5 - 4x - x^2}} \right) + \frac{1}{243} \cdot \frac{(x + 2)^3}{(5 - 4x - x^2)^{\frac{3}{2}}} + C \end{aligned}$$

Fig 3: Ex 3



The triangle used here can be seen in Figure  
Then there are integrals with the format  $\sqrt{a^2 + x^2}$ . Here, the following will be used:

$$x = a \tan \theta, \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

and:

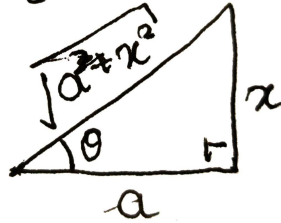
$$dx = a \sec^2 \theta d\theta$$

Therefore,

$$\sqrt{a^2 + x^2} = a |\sec \theta| = a \sec \theta$$

Figure can be seen in .

Fig 4: (b) triangle #



Usage demonstrated:

$$\int_0^2 x^3 \sqrt{x^2 + 4} dx$$

$$x = 2 \tan \theta$$

$$dx = 2 \sec^2 \theta d\theta$$

$$\text{New limits} = \int_0^{\pi/4} 8 \tan^3 \theta (2 \sec \theta) 2 \sec^2 \theta d\theta$$

$$= 32 \int_0^{\pi/4} \tan^3 \theta \sec^3 \theta d\theta$$

$$\text{Peel off sec, \& tan} = 32 \int_0^{\pi/4} \tan^2 \theta \sec^2 \theta \sec \theta \tan \theta d\theta$$

$$= 32 \int_0^{\pi/4} (\sec^2 \theta - 1) \sec^2 \theta \sec \theta \tan \theta d\theta$$

$$u = \sec \theta$$

$$du = \sec \theta \tan \theta d\theta$$

$$\Rightarrow 32 \int_1^{\sqrt{2}} (u^2 - 1) u^2 du$$

$$= 32 \int_1^{\sqrt{2}} u^4 - u^2 du$$

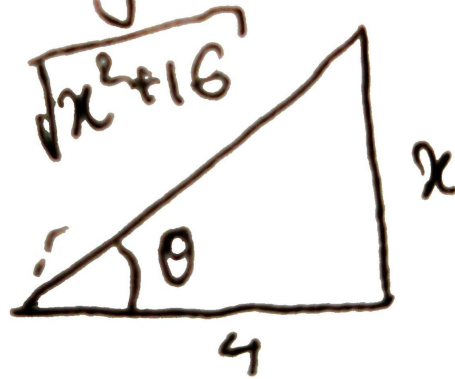
$$= 32 \left[ \frac{u^5}{5} - \frac{u^3}{3} \right]_1^{\sqrt{2}}$$

$$= 32 \left( \frac{4\sqrt{2}}{5} - \frac{2\sqrt{2}}{3} - \left( \frac{1}{5} - \frac{1}{3} \right) \right)$$

$$= 32 \left[ \frac{2\sqrt{2} + 2}{15} \right]$$

An easier example:

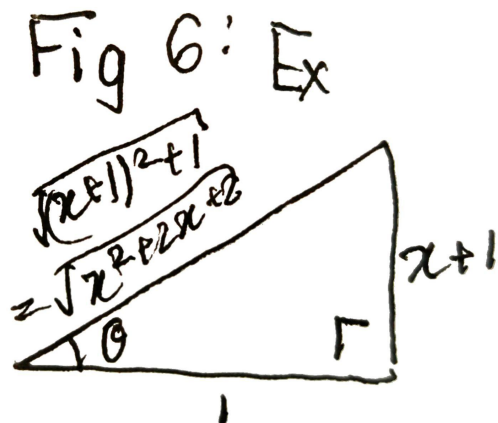
Fig 5: Ex



$$\begin{aligned} \int \frac{1}{\sqrt{x^2 + 16}} dx \\ x &= 4 \tan \theta \\ dx &= 4 \sec^2 \theta d\theta \\ &= \int \frac{4 \sec^2 \theta}{4 \sec \theta} d\theta \\ \text{Memorize} &= \int \sec \theta d\theta \\ &= \ln |\sec \theta + \tan \theta| + C \end{aligned}$$

Don't forget to put it back in terms of  $x$ , by referring to the triangle in Figure  
Then, we can plug  $c$  back into the equation:

$$\begin{aligned} \ln |\sec \theta + \tan \theta| + C &= \ln \left| \frac{\sqrt{x^2 + 16} + x}{4} \right| + C \\ \ln(4) \text{ is constant} &= \ln |\sqrt{x^2 + 16} + x| + C \end{aligned}$$



Another example where we have to complete the square first:

$$\begin{aligned}
 \int \frac{1}{(x^2 + 2x + 2)^2} dx &= \int \frac{1}{[(x^2 + 2x) + 2]^2} dx \\
 &= \int \frac{1}{[(x^2 + 2x + 1) + 1]^2} dx \\
 &= \int \frac{1}{[(x + 1)^2 + 1]^2} dx \\
 x + 1 &= \tan \theta \\
 dx &= \sec^2 \theta d\theta \\
 \int \frac{1}{(x^2 + 2x + 2)^2} dx &= \int \frac{\sec^2 \theta}{(\sec \theta)^4} d\theta \\
 &= \int \cos^2 \theta d\theta \\
 &= \int \frac{1}{2}(1 + \cos 2\theta) d\theta \\
 &= \frac{1}{2}[\theta + \frac{1}{2} \sin 2\theta] + C \\
 &= \frac{1}{2}[\theta + \sin \theta \cos \theta] + C \\
 &= \frac{1}{2} \left( \tan^{-1}(x + 1) + \left( \frac{x + 1}{\sqrt{x^2 + 2x + 2}} \right) \left( \frac{1}{\sqrt{x^2 + 2x + 2}} \right) \right) + C \\
 &= \frac{1}{2} \left( \tan^{-1}(x + 1) + \frac{x + 1}{x^2 + 2x + 2} \right) + C
 \end{aligned}$$

Figure of triangle can be seen in Figure

Finally, there is the integral involving  $\sqrt{x^2 - a^2}$ ,  $a > 0$ . For this,

$$x = a \sec \theta, \theta \in \left[0, \frac{\pi}{2}\right) \text{ or } \left[\pi, \frac{3\pi}{2}\right) \quad dx = a \sec \theta \tan \theta \, d\theta$$

Therefore:

$$\sqrt{x^2 - a^2} = a |\tan \theta| = a \tan \theta$$

Example:

$$\begin{aligned} \int \frac{\sqrt{x^2 - 4}}{x^4} dx \\ x &= 2 \sec \theta \\ dx &= 2 \sec \theta \tan \theta \, d\theta \\ &\Rightarrow \frac{2 \tan \theta}{16 \sec^4 \theta} \cdot 2 \sec \theta \tan \theta \, d\theta \\ &= \frac{1}{4} \int \frac{\tan^2 \theta}{\sec^3 \theta} \, d\theta \\ &= \frac{1}{4} \int \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \cos^3 \theta \, d\theta \\ &= \frac{1}{4} \int \sin^2 \theta \cos \theta \, d\theta \\ u &= \sin \theta \\ du &= \cos \theta \, d\theta \\ &\Rightarrow \frac{1}{4} \int u^2 \, du \\ &= \frac{1}{12} u^3 + C \\ &= \frac{1}{12} \sin^3 \theta + C \\ &= \frac{1}{12} \left( \frac{\sqrt{x^2 - 4}}{x} \right)^3 + C \\ &= \frac{(x^2 - 4)^{\frac{3}{2}}}{12x^3} + C \end{aligned}$$

Another exercise:

$$\int \frac{\sqrt{x^2 - 9}}{x^3} dx$$

$$x = 3 \sec \theta$$

$$dx = 3 \sec \theta \tan \theta d\theta$$

$$= \int \frac{3 \tan \theta}{27 \sec^3 \theta} \cdot 3 \sec \theta \tan \theta d\theta$$

$$= \frac{1}{3} \int \frac{\tan^2 \theta}{\sec^2 \theta} d\theta$$

$$= \frac{1}{3} \int \frac{\sin^2 \theta}{\cos^2 \theta} \cdot \cos^2 \theta d\theta$$

$$= \frac{1}{3} \int \sin^2 \theta d\theta$$

$$\text{Use half angle} = \frac{1}{6} \int (1 - \cos 2\theta) d\theta$$

$$= \frac{1}{6} \left[ \theta - \frac{1}{2} \sin 2\theta \right] + C$$

$$= \frac{1}{6} [\theta - \sin \theta \cos \theta] + C$$

$$= \frac{1}{6} \left[ \sec^{-1} \left( \frac{x}{3} \right) - \left( \frac{x^2 - 9}{x} \right) \left( \frac{3}{x} \right) \right] + C$$

$$= \frac{1}{6} [\sec^{-1}]$$

# Notes

## MATH 152

Turibius Rozario

September 22, 2021

By the next class, we are expected to read 7.5, because that part is just a summary. There will, however, be a reading quiz for it. Today we will focus on 7.4: Partial Fractions.

This is about rational functions, particularly polynomials, where we take one rational function and break it down to multiple. So, if there is a function like:

$$\frac{A(D) + B(C)}{CD} = \frac{A}{C} + \frac{B}{D}$$

Similarly, we can solve integrals by doing that:

$$\begin{aligned} \int \frac{x+5}{x^2+x-2} dx &= \int \left[ \frac{2}{x-1} - \frac{1}{x+2} \right] dx \\ &= 2 \ln|x-1| - \ln|x+2| + C \end{aligned}$$

The hard part is figuring out how to break it down though. The official way for this is "(new broken down function) the partial fraction decomposition of (original function)"

Let  $f(x)$  be any rational function in the form  $f(x) = \frac{P(x)}{Q(x)}$  where both are polynomials. The degree of  $P$  has to be less than the degree of  $Q$ , to do partial fractions. Otherwise, we can do long division to lower the upper value.

For example, for:

$$\int \frac{x^2}{x+4} dx$$

$$(x+4) \overline{)x^2 + 0x + 0}$$

Which equates to

$$\int \left( x - 4 + \frac{16}{x+4} \right) dx = \frac{1}{2}x^2 - 4x + 16 \ln|x+4| + C$$

If  $P$  degree is less than the degree of  $Q$ , then  $P/Q$  can be decomposed to partial fractions, into a first factor of  $O(x)$  as far as possible. Any polynomial  $Q$  can be factored as a linear, such as  $ax + b$ , and irreducible quadratic factors  $ax^2 + bx + c$ . For example, for:

$$Q(x) = x^4 - 16 = (x^2 - 4)(x^2 + 4) = (x - 2)(x + 2)(x^2 + 4)$$

In a way, there are four types of partial fraction decomposition: linear non-repeating, linear repeating, quadratic non-repeating, and quadratic repeating, in order of easiest to highest.

## 1 Nonrepeated linear factor

$ax + b$  will have one term in partial fraction decomposition form  $\frac{A}{ax+b}$  where A is a constant. For example:

$$\int \frac{1}{(t+4)(t-1)} dt = \int \left[ \frac{A}{t+4} + \frac{B}{t-1} \right] dt$$
$$A(t-1) + B(t+4) = 1 \text{ for all } t$$
$$\text{For } t = 1, B = \frac{1}{5}$$
$$\text{For } t = -4, A = -\frac{1}{5}$$

The harder but better in the long run method is to separate the terms by simply multiplying them together, and then separating by terms:

$$At - A + Bt + 4B = 1$$
$$(A + B)t + (-A + 4B) = 0t + 1 \forall t$$
$$\therefore A + B = 0$$
$$-A + 4B = 1$$
$$A = -\frac{1}{5}$$
$$B = \frac{1}{5}$$

And these eventually turn out to be:==

$$\int \frac{1}{(t+4)(t-1)} dt = \frac{1}{5} \int \left[ \frac{1}{t-1} - \frac{1}{t+4} \right] dt$$
$$= \frac{1}{5} \ln \left| \frac{t-1}{t+4} \right| + C$$

Another example:

$$\begin{aligned} \int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6} dx & \text{ do long div} \\ &= \int_0^1 \left( x + 1 + \frac{3x - 4}{x^2 - x - 6} \right) dx \\ &= \int_0^1 \left( x + 1 + \frac{A}{x - 3} + \frac{B}{x + 2} \right) dx \\ 3x - 4 &= A(x + 2) + B(x - 3) \forall x \\ Ax + 2A + Bx - 3B &= 3x - 4 \\ (A + B)x + (2A - 3B) &= 3x - 4 \\ A + B &= 3 \\ 2A - 3B &= -4 \\ \Rightarrow 2A - 3(3 - A) &= -4 \\ 2A + 3A &= -4 + 9 \\ A &= 1 \\ B &= 2 \\ \Rightarrow \int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6} dx &= \int_0^1 \left( x + 1 + \frac{1}{x - 3} + \frac{2}{x + 2} \right) dx \\ &= \left[ \frac{1}{2}x^2 + x + \ln|x - 3| + 2\ln|x + 2| \right]_0^1 \\ &= \frac{3}{2} + \ln \frac{3}{2} \end{aligned}$$

## Repeated linear factor

In the form of

$$(ax + b)^r$$

, where  $r$  is a power greater than 1. This results in a partial fraction decomposition of the following format

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_r}{(ax + b)^r}$$

In case 1, we only had natural log. But here, there is more. Example practice:

$$\int \frac{x^2}{(x-3)(x+2)^2} dx = \int \left[ \frac{A}{x-3} + \frac{B}{x+2} + \frac{C}{(x+2)^2} \right] dx$$

$$A(x+2)^2 + B(x+2)(x-3) + C(x-3) = x^2$$

$$\text{For } x = -2, C = -\frac{4}{5}$$

$$\text{For } x = 3, A = \frac{9}{25}$$

$$\text{For } x = 0, B = \frac{16}{25}$$

$$\int \frac{9/25}{x-3} + \frac{16/25}{x+2} - \frac{4/5}{(x+2)^2} dx = \frac{9}{25} \ln|x-3| + \frac{16}{25} \ln|x+2| - \frac{4}{5} \left( -\frac{1}{x+2} \right) + C$$

Another example:

$$\int \frac{1}{x^2(x-1)^2} dx = \int \left[ \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} \right] dx$$

$$Ax(x-1)^2 + B(x-1)^2 + Cx^2(x-1) + Dx^2 = 1$$

$$\text{For } x = 0, B = 1$$

$$\text{For } x = 1, D = 1$$

$$\text{For } x = 2, 2A + B + 4C + 4D = 1$$

$$2A + 4C = -4$$

$$\text{For } x = -1, -4A + 4B - 2C + D = 1$$

$$-4A - 2C = -4$$

$$\therefore A = 2$$

$$C = -2$$

$$\int \left[ \frac{2}{x} + \frac{1}{x^2} - \frac{2}{x-1} + \frac{1}{(x-1)^2} \right] dx = 2 \ln|x| - \frac{1}{x} - 2 \ln|x-1| - \frac{1}{x-1} + C$$

$$= 2 \ln \left| \frac{x}{x-1} \right| + \frac{1}{x(x-1)} + C$$

## 2 Nonrepeated irreducible quadratic factors

Has form  $ax^2 + bx + c$ , but gets one term. This forces the form:

$$\frac{Ax + B}{ax^2 + bx + c}$$

One specific integral that comes a lot here is:

$$\begin{aligned}\int \frac{1}{x^2 + a^2} dx \\ x = a \tan \theta \\ dx = a \sec^2 \theta d\theta \\ \int \frac{1}{x^2 + a^2} dx &= \int \frac{a \sec^2 \theta}{a^2 \sec^2 \theta} d\theta \\ &= \int \frac{1}{a} d\theta \\ \frac{1}{a} \theta + C &= \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C \\ \int \frac{1}{x^2 + a^2} dx &= \frac{1}{a} \tan^{-1} \left( \frac{x}{a} \right) + C\end{aligned}$$

Note: if  $a$  was 1, the integral would be  $\tan^{-1} x$ .

# Notes

## MATH 152

Turibius Rozario

September 27, 2021

Section 7.6 is not covered in this class. Partial fraction decomposition is used to break down rational functions. The first step is always to divide the top by the bottom, if possible, so that top degree is lower than the bottom. Then simpler rational functions can be integrated.

There are three cases. **Case 1** For nonrepeated linear functions, one term exists:

$$\frac{A}{ax + b} \tag{1}$$

For repeated linear factor, it has  $r$  terms:

$$\frac{A_1}{ax + b} + \frac{A_2}{(ax + b)^2} + \cdots + \frac{A_r}{(ax + b)^r} \tag{2}$$

And then there is non-repeated irreducible quadratic factors:

$$\frac{Ax + B}{ax^2 + bx + c} \tag{3}$$

The following integral comes out a lot, so we should memorize it:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left( \frac{x}{a} + C \right) \tag{4}$$

As an example:

$$\begin{aligned}
 \int \frac{x^2 - x + 6}{x^3 + 3} dx &= \int \frac{x^2 - x + 6}{x(x^2 + 3)} dx && \text{Factor denom.} \\
 \int \left[ \frac{A}{x} + \frac{Bx + C}{x^2 + 3} \right] dx &&& \text{Breakdown} \\
 A(x^2 + 3) + (Bx + C)x &= x^2 - x + 6 && \text{Remove brackets} \\
 Ax^2 + 3A + Bx^2 + Cx &= x^2 - x + 6 && \text{Rearrange} \\
 x^2(A + B) &= 1x^2 && \Rightarrow B = -1 \\
 x(C) &= -1x && \Rightarrow C = -1 \\
 3A &= 6 && \Rightarrow A = 2 \\
 \Rightarrow \int \left[ \frac{2}{x} + \frac{-x - 1}{x^2 + 3} \right] dx &= \int \left[ \frac{2}{x} - \frac{x}{x^2 + 3} - \frac{1}{x^2 + 3} \right] dx && \text{Breakdown} \\
 &= 2 \ln |x| - \frac{1}{2} \ln |x^2 + 3| - \frac{1}{3} \tan^{-1} \left( \frac{x}{3} \right) + C && \text{Integrate}
 \end{aligned}$$

The above example uses the cool  $\tan^{-1}$  integral thingy to get the answer for the middle value. For doing repeated factors:

$$\begin{aligned}
 \int \frac{x^2 - 2x - 1}{(x - 1)^2(x^2 + 1)} dx &= \int \left[ \frac{A}{x - 1} + \frac{B}{(x - 1)^2} + \frac{Cx + D}{x^2 + 1} \right] dx \\
 x^2 - 2x - 1 &= A(x - 1)(x^2 + 1) + B(x^2 + 1) + (Cx + D)(x - 1)^2 \\
 x = 1; B &= -1 \\
 x = 0; -A + B + D &= -1 \Rightarrow A = D = 0 \\
 x = -1; C &= -1 \\
 x = 2; 5A + 5B + 2C + D &= -1 \\
 \Rightarrow D = 1, A = D &= 2 \\
 \int \left[ \frac{1}{x - 1} - \frac{1}{(x - 1)^2} - \frac{x}{x^2 + 1} + \frac{1}{x^2 + 1} \right] dx & \\
 = \ln |x - 1| + \frac{1}{x - 1} - \frac{1}{2} \ln |x^2 + 1| + \tan^{-1} x + C &
 \end{aligned}$$

Case 4 is where repeated quadratic factors are irreducible: =

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \dots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r} \tag{5}$$

# Notes

## MATH 152

Turibius Rozario

October 4, 2021

We will have exam on this Wednesday, from 5:30 to 6:45 in the lecture room, with our Photo ID. Books, notes, calculators/electronic devices are not allowed. No discussion section, but will have Blackboard informbal review session on Wednesday from 4 to 4:50 on Wednesday.

### Improper Integrals

So far, we have seen integrals that are continuous at our bounds, or are just finite.

Occurs in two cases:

1. Infinities are the boundaries.
2. We are integrating the function with a vertical asymptote.

For the first tipe, we will start with the same example as in the text book.

Let  $S$  be region under curve:

$$y = \frac{1}{x^2}, x \geq 1$$

$$\text{Should be } = \int_1^{\infty} \frac{1}{x^2}$$

$$\begin{aligned} \text{Since infinity is just a concept} &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2} \\ &= \lim_{t \rightarrow \infty} - \left[ \frac{1}{x} \right]_1^t \\ &= \lim_{t \rightarrow \infty} - \left( \frac{1}{t} - 1 \right) \\ &= 1 \end{aligned}$$

This is definition of the type 1 of improperi integral. When integrating an improper integral with both limits being infinities, break it down to two an empty. If no limit exits, than we can say that it diverges and the limit DNE. If opposite, then is called convergent.

$$\int_1^{\infty} \frac{1}{x} dx = \lim_{t \rightarrow \infty} \ln |x|_1^t$$

$$= \infty$$

Although both functions  $\frac{1}{x^2}$  and  $\frac{1}{x}$  send to 0, we have had different interpretations of the periodic table.

$\int_1^{\infty} \frac{1}{x^p} dx$  convergence/divergence.

Type 2 improper integrals are if we are doing the integral between two finite boundaries, but there is an asymptote.

1. If  $f$  is continuous on  $[a, b)$ , and discontinuous on  $b$ , then it would be integrated with the limit:

$$\int_a^b f(x) dx = \lim_{t \rightarrow b^-} \int_a^t f(x) dx$$

2. If  $f$  is continuous on  $(a, b]$ , but discontinuous at  $a$ , then a similar thing like (1) will be done.
3. If  $f$  is discontinuous at  $c \ni c \in [a, b]$ , and  $\int_a^c f(x) dx$  and  $\int_c^b f(x) dx$  converges, then:

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx$$

For example:

$$\int_0^1 \frac{1}{x^2} dx = \lim_{t \rightarrow 0^+} \int_t^1 \frac{1}{x^2} dx$$

$$= \lim_{t \rightarrow 0^+} - \left[ \frac{1}{x} \right]_t^1$$

$$= \lim_{t \rightarrow 0^+} \left( -1 + \frac{1}{t} \right)$$

$$= \infty$$

Another example:

$$\int_1^9 \frac{1}{\sqrt[3]{x-9}} dx = \lim_{t \rightarrow 9^-} \int_1^t (x-9)^{-\frac{1}{3}} dx$$

$$= \lim_{t \rightarrow 9^-} \frac{3}{2} (x-9)^{\frac{2}{3}} \Big|_1^t$$

$$= \lim_{t \rightarrow 9^-} \frac{3}{2} \left[ (t-9)^{\frac{2}{3}} - (-8)^{\frac{2}{3}} \right]$$

$$= -6$$

Another example:

$$\begin{aligned}\int_0^4 \frac{1}{x^2 + x - 6} dx &= \int_0^4 \frac{1}{(x-2)(x+3)} dx \\ &= \lim_{a \rightarrow 2^-} \int_0^a \frac{1}{(x-2)(x+3)} dx + \lim_{b \rightarrow 2^+} \int_b^4 \frac{1}{(x-2)(x+3)} dx \\ \lim_{a \rightarrow 2^-} \int_0^a \frac{1}{(x-2)(x+3)} dx &= \lim_{a \rightarrow 2^-} \left[ \frac{1}{5} \ln |x-2| + \frac{1}{5} \ln |x+3| \right]_0^a \\ \ln 0 &= -\infty \\ \ln \infty &= \infty\end{aligned}$$

## Discussion

# Notes

## MATH 152

Turibius Rozario

October 11, 2021

Section 8.1 is about arc length of the graph of a function. It will be derived by looking at a curve  $y = f(x)$ ,  $x \in [a, b]$ . And once done, it can be converted for other functions too. It will be derived like the area under the curve. And approximating via a rectangle.

First, we divide the region  $[a, b]$  into sub intervals:

$$[x_0, x_1], [x_1, x_2], \dots, [x_{i-1}, x_i], \dots, [x_{n-1}, x_n]$$

Each have equal width  $\Delta x$ . For each  $i$ , let  $y_i = f(x_i)$ . That gives us  $n + 1$  points on the curve. There is no if the function is negative. There is a point:  $P_i = (x_i, f(x_i))$ . Length of curve would be approximately equal to the length of the line segments:

$$L \approx \sum_{i=1}^n |P_{i-1}P_i|$$

The length of line segment is  $|P_{i-1}P_i|$ . As  $n \rightarrow \infty$ , exact length of curve can be found.

While the  $\Delta x$  is constant, the  $\Delta y_i$  is not:  $\Delta y_i = y_i - y_{i-1}$ .

$$|P_{i-1}P_i| = \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2} = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

Now, by the MVT,

$$\exists x_i^* \in [x_{i-1}, x_i] \ni f'(x_i^*) = \frac{y_i - y_{i-1}}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x}$$

$$\therefore |P_{i-1}P_i| = \sqrt{(\Delta x)^2 + [f'(x_i^*)]^2 (\Delta x)^2} = \sqrt{1 + [f'(x_i^*)]^2} \Delta x \therefore L = \int_a^b \sqrt{1 + [f'(x_i^*)]^2} dx$$

Note that  $f'$  has to be continuous!

### Exercise 1

$$\begin{aligned}y &= \frac{x^2}{2} - \frac{\ln x}{4} \quad \forall x \in [2, 4] \\f'(x) &= x - \frac{1}{4x} \\ \sqrt{1 + \left(x - \frac{1}{4x}\right)^2} &= \sqrt{1 + x^2 - \frac{1}{2} + \frac{1}{16x^2}} \\ \sqrt{\left(x + \frac{1}{4x}\right)^2} &= \left|x + \frac{1}{4x}\right| \\ &= x + \frac{1}{4x} \quad \forall f(x) > 0, x \in [2, 4] \\ L &= \int_2^4 \left(x + \frac{1}{4x}\right) dx \\ \left[\frac{1}{2}x^2 + \frac{1}{4} \ln |x|\right]_2^4 &= 6 + \frac{1}{4} \ln 2\end{aligned}$$

### Exercise 2

$$\begin{aligned}y &= \ln(\cos x), \quad x \in \left[0, \frac{\pi}{3}\right] \\f'(x) &= -\tan x \\ \sqrt{1 + [f'(x)]^2} &= \sqrt{1 + \tan^2 x} \\ \sqrt{\sec^2 x} &= \sec x \\ L &= \int_0^{\frac{\pi}{3}} \sec x \, dx \\ &= [\ln |\sec x + \tan x|]_0^{\frac{\pi}{3}} \\ &= \ln(2 + \sqrt{3})\end{aligned}$$

### Exercise 3

Arc length functions are functions designed to find area for any given  $x$  value. E.g. from  $(a, f(a))$  to any point  $(x, f(x))$ . The function is denoted as:

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2} \, dt$$

So, for the following:

$$y = \frac{1}{3}x^3 + \frac{1}{4x} \quad \ni x > 0$$

And that the curve is starting in point  $(1, \frac{7}{12})$  to:  $(2, \frac{67}{24})$  and  $(3, \frac{109}{12})$ . The solution:

$$\begin{aligned}
 f'(x) = x^2 - \frac{1}{4x^2} &\Rightarrow \sqrt{1 + (x^2 - \frac{1}{4x^2})^2} \\
 &= \sqrt{(x^2 + \frac{1}{4x^2})^2} \\
 s(x) &= \int_1^x (t^2 + \frac{1}{4t^2}) dt \\
 &= \left[ \frac{t^3}{3} - \frac{1}{4t} \right]_1^x \\
 &= \frac{x^3}{3} - \frac{1}{4x} - \frac{1}{12} \\
 \therefore s(2) &= \frac{59}{24} \\
 \therefore s(3) &= \frac{53}{6}
 \end{aligned}$$

If we wanted to find the distance between  $x = 2$  and  $x = 3$ , we can do  $s(3) - s(2)$ .

## 8.2: Surface Area of Surface of Revolution

A surface of revolution is obtained by revolving a curve about a line. Surface area of a cone can be found with base radius  $r$ , and the slant height  $l$ .

We will have to cut the cone along its slant, and then the radius of the sector would be  $l$ , while the arc length would be  $2\pi r$ . Hence, the area (excluding the bottom. ) can be easily found by:

$$A = \frac{2\pi r}{2\pi l} \times \pi l^2 = \pi r l$$

If we only had a *bit* of the cone, not the entire thing, we can subtract the bigger cone area with the smaller cone area:

Surface area = area of top cone - area of bottom cone

Which results in:

$$\pi r_2(l + l_1) - \pi r_1 l_1 \pi [(r_2 - r_1)l_1 + r_2 l]$$

Since we do not want  $l_1$  because we do not know it, we can convert it into other terms:

$$\begin{aligned}
 \frac{l_1}{r_1} &= \frac{l + l_1}{r_2} \\
 r_2 l_1 &= r_1 l + r_1 l_1 \\
 (r_2 - r_1) l_1 &= r_1 l
 \end{aligned}$$

We can substitute, to get surface area:

$$\begin{aligned}SA &= \pi[r_1l + r_2l] \\ &= \pi(r_1 + r_2)l \\ &= 2\pi\left(\frac{r_1 + r_2}{2}\right)l \\ \therefore SA &= 2\pi rl\end{aligned}$$

Where  $r$  is the average radius.

## Discussion

Given:

$$\begin{aligned}\int_0^3 (x^2 + 3) dx; n = 6 \\ L_6 &= \sum_{i=1}^6 f(x_{i-1})\Delta x \\ &= \Delta x \left[ f(0) + f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) \right] \\ &= \frac{1}{2} \left[ 3 + \frac{13}{4} + 4 + \frac{21}{4} + 7 + \frac{37}{4} \right] \\ &= \frac{1}{2} \left[ 14 + \frac{71}{4} \right] \\ R &= \sum_{i=1}^6 f(x_i)\Delta x \\ &= \frac{1}{2} \left[ f\left(\frac{1}{2}\right) + f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3) \right]\end{aligned}$$

Another exercise:

$$\int_0^2 \frac{e^x}{1+x^2} dx; n = 10; \Delta x = \frac{1}{5}$$

Error bound formulas:

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \ni K = |f''(x)_{max}| \quad (1)$$

$$|E_A| \leq \frac{K(b-a)^3}{24n^2} \quad (2)$$

# Notes

## MATH 152

Turibius Rozario

October 18, 2021

Last time, we discussed how a sequence is an *ordered* list:  $\{a_n\} = \{a_1, a_2, a_3, \dots, a_n, \dots\}$ . The limit of a sequence:  $\lim_{n \rightarrow \infty} a_n = L$ , or more straight forward:  $a_n \rightarrow L$ .  $L$  is the constant that the sequence approaches.

We will go to the mechanics of limits. And discuss the properties of certain sequences. Two things that'll also be discussed: Sequence being monotonic vs bounded.

If we have two sequences that both converge, such as  $a_n \rightarrow A$  and  $b_n \rightarrow B$ , and if  $c$  is any constant, then we can take limits term by term, and take out any constants:

1.

$$\lim_{n \rightarrow \infty} (a_n + b_n) = A + B$$

2.

$$\lim_{n \rightarrow \infty} (a_n - b_n) = A - B$$

3.

$$\lim_{n \rightarrow \infty} c \cdot a_n = c \cdot A$$

4.

$$\lim_{n \rightarrow \infty} c = c$$

5.

$$\lim_{n \rightarrow \infty} (a_n \cdot b_n) = A \cdot B$$

6.

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \frac{A}{B} \text{ (if } B \neq 0\text{)}$$

7.

$$\lim_{n \rightarrow \infty} a_n^p = A^p \text{ for any } p > 0, \text{ provided that } a_n \geq 0$$

8. Squeeze theorem: When we have three sequence, and one is squeezed between two converging series, that one converges to the same value at the point of squeeze:

If, for some integer  $n_0$ , we have  $a_n \leq b_n \leq c_n$  for all  $n \geq n_0$ , and if  $a_n \rightarrow L$  and  $c_n \rightarrow L$ , then  $b_n \rightarrow L$  as  $n \rightarrow \infty$ .

9. If  $\lim_{n \rightarrow \infty} |a_n| = 0$ , then  $\lim_{n \rightarrow \infty} a_n = 0$ . Derived from squeeze theorem:  $-|a_n| \leq a_n \leq |a_n|$  for all  $n$ .

Example problem: Showing that  $\lim_{n \rightarrow \infty} \frac{n}{n+1} = 1$ :

$$\lim_{n \rightarrow \infty} \frac{n}{n+1} = \lim_{n \rightarrow \infty} \frac{\frac{n}{n}}{\frac{n}{n} + \frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{1}{1 + \frac{1}{n}}; \lim_{n \rightarrow \infty} \frac{1}{n} = 0 \therefore \frac{\lim_{n \rightarrow \infty} 1}{\lim_{n \rightarrow \infty} 1 + \lim_{n \rightarrow \infty} \frac{1}{n}} = \frac{1}{1+0} = 1$$

This can be alternatively shown by:

$$\frac{n}{n+1} \approx \frac{n}{n} = 1, \text{ as } n \rightarrow \infty$$

Example of limit law number 9, that if  $|a_n| \rightarrow 0$ , then  $a_n \rightarrow 0$ :

$$\left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \rightarrow 0$$

Another example of this:

$$\left| \frac{(-1)^2 n}{n+1} \right| = \frac{n}{n+1}$$

, therefore  $\left\{ \frac{(-1)^n n}{n+1} \right\}$

Issue arises when we use trigonometric function:  $0 \leq \left| \frac{\sin 2n}{1+\sqrt{n}} \right| \leq \frac{1}{1+\sqrt{n}} \rightarrow 0$ .

Example exercise problem: Finding the derivative of a sequence, or calling it diverging:

$$a_n = \frac{n+1}{3n-1} = \frac{1 + \frac{1}{n}}{3 - \frac{1}{n}} \rightarrow \frac{1}{3}$$

Another:

$$a_n = \frac{n}{1 + \sqrt{n}} = \frac{\sqrt{n}}{\frac{1}{\sqrt{n}} + 1} = \frac{\infty}{1} = \text{Diverges}$$

Another:

$$a_n = \cos \frac{2}{n} \rightarrow \cos 0 \rightarrow 1$$

In certain cases, we can use L'Hospital's Rule:

$$a_n = \frac{\ln n}{\ln 2n} \rightarrow \frac{\infty}{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{\ln n}{\ln 2n} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x}}{\frac{2}{2x}} = \lim_{x \rightarrow \infty} 1 = 1$$

$$\therefore a_n \rightarrow 1$$

Special type of sequence is the Geometric Sequence. In its most basic form, it looks like  $\{r^n\} = \{r, r^2, r^3, \dots\}$ , and in full form:  $\{a \cdot r^n\}$ . This converges if  $-1 < r \leq 1$ , and diverges otherwise.

$$\lim_{n \rightarrow \infty} r^n = 0 \iff -1 < r < 1, 1 \iff r = 1$$

Some tactics when doing these:

$$\lim_{n \rightarrow \infty} \frac{3^{2n}}{5^n} = \lim_{n \rightarrow \infty} \left( \frac{9}{5} \right)^n = \infty$$

Another:

$$\lim_{n \rightarrow \infty} \frac{3^{n+1}}{2^{2n+3}} = \lim_{n \rightarrow \infty} \frac{3 \times 3^n}{8 \times 4^n} = \frac{3}{8} \lim_{n \rightarrow \infty} \left( \frac{3}{4} \right)^n = 0$$

What does it mean for a sequence to be increasing or decreasing? If it is either increasing or decreasing, it is known as monotonic. More formal definition:

1. Sequence  $\{a_n\}$  is increasing if  $a_n < a_{n+1} \forall n$ .
2. Sequence  $\{a_n\}$  is decreasing if  $a_n > a_{n+1} \forall n$ .
3. Sequence  $\{a_n\}$  is monotonic if it is either increasing or decreasing.

We can show that a sequence such as  $\{\frac{n}{n+1}\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$  is increasing in the following ways:

$$\begin{aligned} a_{n+1} - a_n &= \frac{n+1}{n+2} - \frac{n}{n+1} \\ \frac{(n+1)^2 - n(n+2)}{(n+2)(n+1)} &= \frac{n^2 + 2n + 1 - n^2 - 2n}{(n+2)(n+1)} \\ \frac{1}{(n+2)(n+1)} &> 0, \therefore a_{n+1} > a_n \end{aligned}$$

The above would be decreasing if it was less than 0.

Alternatively, we can show the ratio:

$$\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{n+2}}{\frac{n}{n+1}} = \frac{n+1}{n+2} \cdot \frac{n+1}{n} = \frac{n^2 + 2n + 1}{n^2 + 2n} > 1, \therefore a_{n+1} > a_n$$

This would be decreasing if it was less than 1.

And the final method is to use a function:

Let  $f(x) = \frac{x}{x+1}$ . Then:

$$f'(x) = \frac{(x+1)(1) - x(1)}{(x+1)^2} = \frac{x+1-x}{(x+1)^2} = \frac{1}{(x+1)^2} > 0, \forall x \geq 1$$

Therefore, the sequence is increasing for  $n \geq 1$ .

Definition for bounded:

1.  $\{a_n\}$  is bounded above if  $\exists M \ni a_n \leq M, \forall n$ .
2.  $\{a_n\}$  is bounded below if  $\exists m \ni a_n \geq m, \forall n$ .
3.  $\{a_n\}$  is a bounded sequence if it is bounded both above and below:  $m \leq a_n \leq M, \forall n$

So, for example,  $\{\frac{n}{n+1}\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$  is bounded because  $0 < \frac{n}{n+1} < 1, \forall n \geq 1$ . The greatest lower bound, however, is  $\frac{1}{2}$ . This sequence is both monotonic and bounded.

Is the sequence:

1. increasing, decreasing, or not monotonic?
2. bounded?

Sequences:

1.  $a_n = \frac{1}{5^n}; a_{n+1} = \frac{1}{5^{n+1}} = \frac{1}{5} \times \frac{1}{5^n} < a_n$ , therefore is decreasing. And  $0 < a_n \leq \frac{1}{5}$ , hence bounded.

2.  $a_n = \cos\left(\frac{\pi}{2}\right)$  cycles between 0, -1, and 1, hence is not monotonic. This is bounded:  $-1 \leq a_n \leq 1$ .
3.  $a_n = n + \frac{1}{n} \Rightarrow \left\{2, 2\frac{1}{2}, 3\frac{1}{3}, 4\frac{1}{4}, \dots\right\}$ . For this, it is easiest to use a function to prove that this is increasing (monotonic):  $f(x) = x + \frac{1}{x} \Rightarrow f'(x) = 1 - \frac{1}{x^2} > 0, \forall x > 1$ , therefore is increasing. It is bounded below at 2, but not bounded above, hence is overall not bounded, since  $a_n \rightarrow \infty$ .

# Notes

## MATH 152

Turibius Rozario

October 20, 2021

Adding terms of sequence is called an infinite series, or more generally, a series. Denoted by:

$$\sum_{n=1}^{\infty} a_n \text{ or } \sum a_n$$

If we have a sequence of partial sums:

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^n a_i \\ &\vdots \end{aligned}$$

The  $n$ th partial sum of a sequence is denoted as  $\{s_n\}$ . If  $\{s_n\}$  converges, and  $\lim_{n \rightarrow \infty} s_n = s$ , then  $\sum a_n$  converges:

$$\sum_{n=1}^{\infty} a_n = s$$

$s$  is the sum of the series. If the sequence of sums diverges, then the series of the sequence  $\{a_n\}$  also diverges.

One series that diverges is

$$\sum_{n=1}^{\infty} 1 = 1 + 1 + 1 + \cdots + 1$$

. The sequence of series would be divergent:  $s_n = \sum_{n=1}^{\infty} 1 = n \rightarrow \infty$ .

Geometric series is a series:  $a + ar + ar^2 + ar^3 + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$ .  $a$  cannot be 0. If there was only  $a$  for the series, it would go to the infinities (such as ) depending on whether  $a$  or  $b$  defeated you by better than you as usual.

$$\begin{aligned} s_n &= a + ar + ar^2 + \cdots + ar^{n-1} \\ s_n &= a + ar + ar^2 + \cdots + ar^{n-1} + a \end{aligned}$$

# Notes

## MATH 152

Turibius Rozario

October 25, 2021

Last time we did integral test. If we have a function  $f(x)$  is positive, continuous, and decreasing on  $[1, \infty)$ . If  $a_n = f(n)$ , then

$$\sum_{n=1}^{\infty} a_n \text{ converges} \iff \int_1^{\infty} f(x) dx \text{ converges}$$

Example:

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + 4}$$

Since the function  $f(x) = \frac{1}{x^2+4}$  is positive, continuous, and decreasing on  $[1, \infty)$ , the Integral Test applies.

$$\begin{aligned} \int_1^{\infty} \frac{1}{x^2 + 4} dx &= \lim_{t \rightarrow \infty} \int_1^t \frac{1}{x^2 + 4} dx \\ \lim_{t \rightarrow \infty} \frac{1}{2} \left[ \tan^{-1} \left( \frac{x}{2} \right) \right]_1^t &= \frac{1}{2} \lim_{t \rightarrow \infty} \left[ \tan^{-1} \left( \frac{t}{2} \right) - \tan^{-1} \left( \frac{1}{2} \right) \right] \\ \frac{1}{2} \left[ \frac{\pi}{2} - \tan^{-1} \left( \frac{1}{2} \right) \right] \end{aligned}$$

Since the limit of the integral is finite, the series converges by the Integral Test.

### Comparison Test

This is commonly applied in presence of series similar to a geometric series or a p-series. Take the example  $\sum_{n=1}^{\infty} \frac{1}{2^n+1}$ . When  $n$  is large, it is similar to  $\frac{1}{2^n}$ , and since the latter converges, we can use Direct Comparison Test:

$$\begin{aligned} 2^n + 1 &> 2^n \\ \frac{1}{2^n + 1} &< \frac{1}{2^n} \forall n \in \mathbb{N} \\ 0 < \frac{1}{2^n + 1} < \frac{1}{2^n}, \forall n \geq 1 &\implies 0 < \sum_{n=1}^{\infty} \frac{1}{2^n + 1} < \sum_{n=1}^{\infty} \frac{1}{2^n} \end{aligned}$$

For other functions like

$$\frac{n}{n^2 - n - 1} \approx \frac{n}{n^2} = \frac{1}{n}$$

# Notes

## MATH 152

Turibius Rozario

October 27, 2021

Chapter 11.7 is just a summary. We should read it by next time. We will do 11.5 and first half of 11.6.

Sub of converging alternating series lies between any two consecutive partial sums  $s_n$  and  $s_{n+1}$ .

$$R_n = s - s_n \leq b_{n+1} \tag{1}$$

For questions about the terms  $n$  we need, we can

# Notes

## MATH 152

Turibius Rozario

November 1, 2021

Exam will be here. Since exam 1. So units 7.7-8, 8.1-8.2, 11.1-7. Bring ID. No notes/book/calculators allowed. Zainab will hold informal review session at 4 to 4:50. No discussion.

### Root Test

Given a series  $\Sigma a_n$ , suppose

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = L$$

This would result in:

1. If  $L < 1$ , then absolutely convergent
2. If  $L > 1$ , then divergent
3. If  $L = 1$ , then inconclusive.

This works because when it approaches infinity:

$$\sqrt[n]{|a_n|} \approx L|a_n| \approx L^n$$

So the sequence  $\Sigma |a_n|$  is similar to geometric series with  $r = L$ .

Example:

$$\sum_{n=1}^{\infty} \frac{n(-3)^n}{4^{n-1}} = \sum_{n=1}^{\infty} 4n \left(-\frac{3}{4}\right)^n$$

$$\sqrt[n]{|a_n|} = \left[4n \left(\frac{3}{4}\right)^n\right]^{\frac{1}{n}}$$

$$4^{\frac{1}{n}} n^{\frac{1}{n}} \left(\frac{3}{4}\right) = 1$$

$$4^0 = 1$$

$$\therefore \lim_{n \rightarrow \infty} n^{\frac{1}{4}} = \frac{3}{4} < 1$$

In the exam, mention which test you are using! Root test cannot be used on ratio test or 'n!'.

## Power Series

Power series converges where domain is set of all  $x$  for which series converges. There is no such power series that diverges. If  $x \in (s1,1)$ , it converges to the function  $\frac{1}{1-x}$ . They converge at  $a$  always, but sometimes also at other points.

Three possibilities of convergence for power series:

1. Convergence for  $x = a$ , hence  $R = 0$ .

# Notes

## MATH 152

Turibius Rozario

November 8, 2021

Today's topic is on representing functions as power series. We know:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots = \sum_{n=0}^{\infty} x^n, |x| < 1$$

We can use this to express similar functions. For example:

$$f(x) = \frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n, \forall |x| < 1$$

We know that the interval of convergence is definitely  $(-1, 1)$ , but we normally have to test endpoints. For this, we don't because  $x^n$  never approaches 0.

Another example:

$$f(x) = \frac{3}{1-x^4} = 3 \times \frac{1}{1-x^4} = 3 \sum_{n=0}^{\infty} (x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}$$

This would converge for  $|x^4| < 1 \Rightarrow |x| < 1$ .

We can just treat other variables separately:

$$f(x) = \frac{x}{1+4x} = x \cdot \frac{1}{1-(-4x)} = x \sum_{n=0}^{\infty} (-4x)^n = \sum_{n=0}^{\infty} (-1)^n 4^n x^{n+1}$$

Another example:

$$f(x) = \frac{x^2}{3-x} = \frac{x^2}{3} \times \frac{1}{1-\left(\frac{x}{3}\right)} = \frac{x^2}{3} \sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{x^{n+2}}{3^{n+1}}$$

The radius of convergence would be 3.

If we have a power series that is

$$\sum c_n(x-a)^n$$

and it has a radius of convergence of  $R > 0$ , then

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + \cdots = \sum_{n=0}^{\infty} c_n(x - a)^n$$

This would be differentiable for  $(a - R, a + R)$

# Notes

## MATH 152

Turibius Rozario

November 10, 2021

Taylor and Maclaurin series Theorem:

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n, |x-a| < R$$

then its coefficients are:

$$c_n = \frac{f^{(n)}(a)}{n!}$$

So, we get:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

Which would equal to:

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \dots$$

This is called a 'Taylor series of  $f$  at  $a$ ' (or centered at  $a$ ).

The first term must be  $f(a)$  so that we get the constant height when all  $x$  is at  $a$ , so that all other terms become 0.

Maclaurin series is a special case where  $a = 0$ :

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Today we are going to focus on finding the Taylor/Maclaurin series of many functions.

The Taylor series to some  $n$  gives us an approximation. For example,  $T_2$  approximates the function and has  $f(a)$  and  $f'(a)$ , which would be a linear approximation. This can be seen in unit 11.11.

Some important Maclaurin series:

1.

$$f(x) = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \forall x \in (-1, 1)$$

2.  $f(x) = \tan^{-1} x$  can have its Maclaurin series derived by the fact that inverse tangent is:

$$\begin{aligned} \frac{d}{dx}[\tan^{-1} x] &= \frac{1}{1+x^2} \\ \Rightarrow \tan^{-1} x &= \int \frac{1}{1+x^2} \\ \int \frac{1}{1-(-x^2)} dx &= \int \sum_{n=0}^{\infty} (-x^2)^n dx \\ \int \sum_{n=0}^{\infty} (-1)^n x^{2n} dx &= C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} \end{aligned}$$

This is true for  $|-x^2| < 1$ , which means it has a radius of  $R = 1$ . If we plug in the end points to see the interval of convergence. First, to find  $C$ :

$$\begin{aligned} \tan^{-1} 0 &= C + \sum_{n=0}^{\infty} 0 \\ C &= \tan^{-1} 0 = 0 \end{aligned}$$

$$\begin{aligned} x = -1 : \sum \frac{(-1)^n(-1)}{2n+1} &= \sum \frac{(-1)^{n+1}}{2n+1} \text{ which converges} \\ x = 1 : \sum \frac{(-1)^n}{2n+1} &\text{ which converges} \end{aligned}$$

Therefore:

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + \frac{(-1)^n x^{2n+1}}{2n+1}$$

3.  $f(x) = e^x$  is easier because no matter how many derivatives we have, we end up having the same derivative. For example:  $f(0) = f'(0) = f''(0) = \dots = f^{(n)}(0) = 1, \forall n \geq 0$  Therefore:

$$e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

To find the radius and interval of convergence, we can use the root test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{(n+1)!} \times \frac{n!}{x^n} \right| = \frac{|x|}{n+1} \rightarrow 0 < 1 \forall x \in \mathbb{R}$$

Note that this series goes to  $e$  very quickly.

4. sin and cos works in a similar way. To find the Maclaurin series for the former:

Firstly, we know that the derivative of sin repeats at 0:  $f(x) = 0, f'(x) = 1, f''(0) = 0, f'''(0) = -1, \dots$

Now, if we write these out:

$$\sin x = \frac{0}{0!} + \frac{1}{1!}x + \frac{0}{2!}x^2 - \frac{1}{3!}x^3 + \frac{0}{4!}x^4 + \frac{1}{5!}x^5 + \frac{0}{6!}x^6 - \frac{1}{7!}x^7 + \dots$$

Which can be rewritten as:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}x^{2n+1}}{(2n+1)!}$$

We can use the ratio test to figure out the radius and interval of convergence, which would be  $R = \infty$  and  $I : (-\infty, \infty)$ .

Notice how similar it is to  $\tan^{-1}$ . Both this and  $\sin x$  are known as odd functions.  $\cos$ , on the contrary, is similar but an even function.

5. Now, for  $f(x) = \cos x$ . First, we can do the same thing as  $\sin$  to find its derivatives:  $f(0) = 1, f'(0) = 0, f''(0) = -1, f'''(0) = 0, \dots$

Therefore:

$$\cos x = \frac{1}{0!} + \frac{0}{1!}x - \frac{1}{2!}x^2 + \frac{0}{3!}x^3 + \frac{1}{4!}x^4 + \dots = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}x^{2n}}{(2n)!}, \forall x \in (-\infty, \infty)$$

Note that the angles are to be in radians, not degrees.

# Notes

## MATH 152

Turibius Rozario

November 15, 2021

Quick recap:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots \quad (1)$$

And a few important MacLaurin series:

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \ni x \in (-1, 1) \quad (2)$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots \ni x \in [-1, 1] \quad (3)$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \ni x \in (-\infty, \infty) \quad (4)$$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \ni x \in (-\infty, \infty) \quad (5)$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \ni x \in (-\infty, \infty) \quad (6)$$

There is also the binomial series, which is in the format  $(1+x)^k \ni k \in \mathbb{R}$  that will be touched on later today.

$$\begin{aligned} f(0) &= 1 \\ f'(0) &= k(1+x)^{k-1} && = k \\ f''(0) &= k(k-1)(1+x)^{k-2} && = k(k-1) \\ f'''(0) &= k(k-1)(k-2)(1+x)^{k-3} && = k(k-1)(k-2) \\ &\vdots \\ f^{(n)}(0) &= k(k-1)\dots(k-n+1) \end{aligned}$$

Therefore, if  $k$  is any real number, the equation is:

$$(1+x)^k = \sum_{n=0}^{\infty} \frac{k(k-1)\dots(k-n+1)}{n!} x^n = \sum_{n=0}^{\infty} \binom{k}{n} x^n \ni R=1, k \in \mathbb{R} \quad (7)$$

As an example:

$$\begin{aligned} f(x) &= \sqrt[3]{1+2x} = (1+2x)^{\frac{1}{3}} \\ (1+2x)^{\frac{1}{3}} &= \sum_{n=0}^{\infty} \binom{\frac{1}{3}}{n} 2^n x^n \\ &= 1 + \frac{1}{3}(2x) + \frac{\frac{1}{3}(-\frac{2}{3})}{2!}(2x)^2 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})}{3!}(2x)^3 + \frac{\frac{1}{3}(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{4!}(2x)^4 \\ &= 1 + \frac{2}{3}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1} 2 \times 5 \times 8 \times \dots \times (3n-4) 2^n}{n! 3^n} x^n, \ni R = \frac{1}{2} \end{aligned}$$

# Notes

## MATH 152

Turibius Rozario

November 17, 2021

Today we will finish section 11.11, and then talk about section 10.5.

On the interval of convergence of the Taylor series,  $T_n(x)$  is the degree of  $n$  polynomial that best approximates  $f(x)$ . The closer  $x$  is to  $a$ , the better the approximation.

$$T_n(x) = \sum_{i=0}^n \frac{f^{(i)}(a)}{i!} (x-a)^i \quad (1)$$

Note that the following happens to be linear approximation:

$$f(x) \approx T_1(x) = f(a) + f'(a)(x-a) \quad (2)$$

### Taylor's Inequality

In general, for any  $n$ , we have:

$$f(x) = T_n(x) + R_n(x)$$

where  $R_n(x)$  is the remainder of the Taylor series, that is, the terms of degree  $n+1$ .

The error would be the absolute value of the remainder. We will use the Taylor polynomial for the approximation of the function. We will go to the next degree, and find its max within an interval:

$$|f^{(n+1)}(x)| \leq M \forall |x-a| \leq d \implies |R_n(x)| \leq \frac{M}{(n+1)!} |x-a|^{n+1} \forall |x-a| \leq d \quad (3)$$

For example: given:

$$f(x) = \frac{1}{x^2}; n = 2, a = 1; x \in [0.9, 1.1]$$

To approximate  $f$  by a Taylor polynomial of degree  $n$  centered at  $a$ :

$$T_2(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2$$

$$f(1) = 1;$$

$$f'(1) = \frac{-2}{x^3} = -2$$

$$f''(1) = \frac{6}{x^4} = 6$$

$$T_2(x) = 1 - 2(x-1) + 3(x-1)^2$$

Next, to find the accuracy of the estimation in the given interval:

$$\begin{aligned} f'''(x) &= -\frac{24}{x^5}; & x \in [0.9, 1.1] \\ |f'''(x)| &= \frac{24}{|x|^5} & \leq \frac{24}{0.9^5} \\ |R_2(x)| &\leq \frac{24}{3!} |x-1|^3 & \leq \frac{24 \times 0.1^3}{0.9^6 \times 3!} \end{aligned}$$

Since the bottom has a factorial, when  $n \rightarrow \infty$ , the remainder  $|R_n(x)| \rightarrow 0$ . Another example:

$$f(x) = \cos x; ; n = 4 \ a = \frac{\pi}{3}; x \in [0, \frac{2\pi}{3}]$$

First, to find the Taylor approximation  $T_4(x)$ :

$$\begin{aligned} f\left(\frac{\pi}{3}\right) &= \frac{1}{2} \\ f'\left(\frac{\pi}{3}\right) &= -\sin x & = -\frac{\sqrt{3}}{2} \\ f''\left(\frac{\pi}{3}\right) &= -\cos x & = -\frac{1}{2} \\ f'''\left(\frac{\pi}{3}\right) &= \sin x & = \frac{\sqrt{3}}{2} \\ f^{(4)}\left(\frac{\pi}{3}\right) &= \cos x & = \frac{1}{2} \end{aligned}$$

Therefore:

$$T_4(x) = \frac{1}{2} - \frac{\sqrt{3}}{2} \left(x - \frac{\pi}{3}\right) - \frac{1}{4} \left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12} \left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48} \left(x - \frac{\pi}{3}\right)^4$$

To find the remainder:

$$\begin{aligned} f^{(5)}(x) &= -\sin x \\ |R_4(x)| &\leq \frac{1}{5!} \left|x - \frac{\pi}{3}\right|^5 & \leq \frac{1}{5!} \left(\frac{\pi}{3}\right)^5 \end{aligned}$$

## Conic Sections

10.5 is on conic sections. 10.1 and .2 are parametric equations, and its mathematical functions. 10.3 and 10.4 are polar equations.

Definition of a Parabola: The set of all points in the plane that are the same distance from a fixed point F (the focus), and a fixed line (the directrix). The vertex of the parabola is the point right between the focus and the vertex.

If the vertex is the origin and the axis is the y-axis, then the equation of the parabola with focus at  $(0, p)$  and directrix  $y = -p$ :

$$x^2 = 4py \tag{4}$$

This parabola opens up and down because  $x$  is squared, and it will open up if  $p > 0$ , and down if  $p < 0$ . The focal diameter is the length passing through the focus and parallel to the directrix, and is equal to  $4p$ .

If we shift it so that the vertex is at  $(h, k)$ , then the equation changes. In that case, the focus will be at  $(h, k + p)$ , and the directrix is  $y = k - p$ , with new equation:

$$(x - h)^2 = 4p(y - k) \quad (5)$$

If the vertex is the origin and the axis is the x-axis instead, then the equation with the parabola with focus  $(p, 0)$  and directrix  $x = -p$ , it would be:

$$y^2 = 4px$$

The curve will open right if  $p > 0$  and left if  $p < 0$  (look at where the less than/greater than symbol is pointing). The vertex here would be:

$$(y - k)^2 = 4p(x - h) \quad (6)$$

And the focus will be  $(h + p, k)$ , and directrix would be  $x = h - p$

Example: Finding the vertex, focus, and the directrix of the parabola, and sketching its graph.

$$(x + 2)^2 = 8(y - 3)$$

$$\text{vertex} = (-2, 3)$$

$$\text{focus} = (-2, 5)$$

$$\text{directrix } y = 1$$

$$\text{width} = |4 \times p| = 8$$

For the sketch, view figure 2.

Another example: Finding the equation of parabola with focus at  $(-4, 0)$  and directrix at  $x = 2$ .

$$\text{vertex} = (2 - 4)/2 \quad = -1$$

$$p = -4 - (-1) \quad = -3$$

$$\therefore y^2 = 4(-3)(x + 1)$$

# Notes

## MATH 152

Turibius Rozario

November 22, 2021

Today we will first finish ellipses and hyperbolas. In the end, talk about parametric.

### Ellipses

An ellipse is a set of all points in the plane for which the sum of the distances from the two fixed points (the 'foci') is equal.

For any ellipse, there are 2 positive constants:  $a \geq b > 0$ . For an ellipse centered at the origin, the equation is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

with the foci located at:

$$(\pm c, 0) \ni c^2 = a^2 - b^2 \quad (2)$$

and the vertices are located at  $(\pm a, 0)$ . The segment joining  $(\pm a, 0)$  is the major axis, and the shorter segment connecting  $(0, \pm b)$  is the minor axis.

Notice that the curve is centered at the origin. If we wanted to center it at the point  $(h, k)$ , then the equation will be:

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1 \quad (3)$$

and foci at  $(h \pm c, k)$ , and vertices at  $(h \pm a, k)$ .

For a vertical ellipse, the equation would therefore be:

$$\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1 \ni a \geq b > 0 \quad (4)$$

and the foci will be at  $(0, \pm c \ni c^2 = a^2 - b^2)$  and vertices at  $(0, \pm a)$ .

For solving equations into the format of these equations, completing the square is the ideal method. For example:

$$\begin{aligned} 9x^2 - 18x + 4y^2 &= 27 \\ 9(x^2 - 2x) + 4y^2 &= 27 \\ 9(x-1)^2 + 4y^2 &= 27 + 9 \\ \frac{(x-1)^2}{4} + \frac{y^2}{9} &= 1 \end{aligned}$$

This ellipse is centered at  $(1,0)$ ,  $a = 3$ ,  $b = 2$ ,  $c^2 = a^2 - b^2 = 9 - 4 = 5 \Rightarrow c = \sqrt{5}$ , and therefore, vertices are at  $(1,3)$  and  $(1, -3)$ , and the foci are at  $(1, \sqrt{5})$ ,  $(1, -\sqrt{5})$ .

## Hyperbolas

A hyperbola is the set of all points in the plane for which the difference of the distances from two fixed points (the 'foci') is constant. The general equation, when centered at origin, is:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad (5)$$

With no limitation on the values for  $a$  and  $b$ . These have the foci at

$$(\pm c, 0) \ni c^2 = a^2 + b^2 \quad (6)$$

And vertices at  $\pm a, 0$  and asymptotes at  $y = \pm \frac{b}{a}x$ .

When the center is changed, the equation becomes:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1 \ni \text{foci} = (h \pm c, k), \text{ vertices} = (h \pm a, k) \quad (7)$$

For a vertical hyperbola:

$$\frac{(y-k)^2}{b^2} - \frac{(x-h)^2}{a^2} = 1 \ni \text{foci} = (h, k \pm c), \text{ vertices} = (h, k \pm a) \quad (8)$$

The asymptote will be located at:

$$y - k = \pm \frac{a}{b}(x - h) \quad (9)$$

## Parametric Curves

Defining  $x$  and  $y$  in terms of a third variable  $t$  — called a parameter — is called a parametric curve:

$$x = f(t), y = g(t) \quad (10)$$

$t$  can have no limits, or can have limits. If it is the latter, where  $a \leq t \leq b$ , then the points  $(f(a), g(a))$  is the initial point, and  $(f(b), g(b))$  is the terminal point of the curve  $C$ .

The parameter,  $t$ , can be removed in certain cases to produce a simpler Cartesian equation.

# Notes

## MATH 152

Turibius Rozario

November 29, 2021

Polar coordinates are a system of coordinate system represented by  $(r, \theta)$  where  $r$  is the distance from the origin, and  $\theta$  is the angle starting off with the polar axis (Cartesian equivalent: positive x-axis). The origin is called the pole.  $r$  can be both positive and negative. This system is useful for certain equations. There's always infinitely infinite representation of point  $P$ , since the radius can be negative, and  $\theta$  can change.

Polar coordinates can be converted to cartesian by using:

$$x = r \cos \theta; y = r \sin \theta \quad (1)$$

Cartesian coordinates to polar by:

$$r^2 = x^2 + y^2; \tan \theta = \frac{y}{x} \quad (2)$$

Polar curves are represented as:

$$r = f(\theta) \quad (3)$$

Polar coordinates can have multiple representations. The equation  $F(r, \theta) = 0$  should have all points that have at least one representation  $(r, \theta)$ .

Exercise: To convert  $r = \sqrt{5}$  into a Cartesian equation, we know that  $r^2 = 5$ , therefore  $x^2 + y^2 = 5$ .

To convert  $r = 2 \cos \theta$  to Cartesian, we do this:

$$\begin{aligned} r^2 &= 2r \cos \theta \\ x^2 + y^2 &= 2x \\ \Rightarrow x^2 - 2x + y^2 &= 0 \\ \Rightarrow x^2 - 2x + 1 + y^2 &= 1 \\ \therefore (x - 1)^2 + y^2 &= 1 \end{aligned}$$

# Notes

## MATH 152

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We can plot a polar curve with a Cartesian-like graph, with  $r$  as the  $y$ -axis, and  $\theta$  as the  $x$ -axis. General equation for flower like graphs:

$$r = a \sin(k\theta) \tag{1}$$

$$r = a \cos(k\theta) \tag{2}$$

where  $a$  is any constant (controlling how big the graph is) and  $k$  is a positive integer:

1.  $k$  leaves if  $k$  is odd.
2.  $2k$  leaves if  $k$  is even.